STABILITY OF TWO-LAYER STRATIFIED FLOW DOWN AN INCLINED PLANE

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ABSTRACT

The stability of flow down an inclined plane has been investigated for the case of a stratified fluid system consisting of two layers of viscous fluid of different densities. This problem is an extension of the works of Benjamin and Yih for a homogeneous fluid; thus their results are a special case of the solution for this more general problem. Asymptotic cases for long and short wave-length disturbances are considered, and the neutral stability curve is estimated. Reynolds numbers for the bifurcation point of the neutral curve are found for various ratios of density and depth of the two layers. For long waves, shear wave instability is also studied and is found to be damped. It is found that the addition of another film of fluid of lighter density over the original film destabilizes the original free surface disturbances.

It is hoped that this work will bear on problems of film flow stabilizing techniques, and will also be of interest in the study of the stability of undercurrents in reservoirs.

1. INTRODUCTION

The investigation of the stability of laminar flow of a homogeneous fluid down an inclined plane has been undertaken by Kapitza (1948, 1949), Yih (1955a), and Benjamin (1957), and was recently given a definitive treatment by Yih (1963). Yih's results showed that for long waves (small α), $R = \frac{5}{6} \cot \theta$ is the critical Reynolds number above which some disturbances will be amplified, and the line $\alpha = 0$ in the α - R plane is part of the neutral stability curve, and that very short waves are damped by surface tension.

In this report, the problem has been extended to flow of a heterogeneous system consisting of two layers of viscous fluid of different densities. The superposition of a lighter fluid on top of a heavier fluid introduces a fluid-fluid interface. The question then arises as to what effects the presence of the upper fluid and the interface have on the hydrodynamic stability of the system. These effects will be examined with respect to both surface disturbances and shear waves.

This study is of interest to various problems of film flows of two liquids that occur in many industrial processes. It also sheds light to the problem of the initiation of mixing of density currents that flow into the reservoir from catchment areas.

2. THE BASIC FLOW

In this section the basic unperturbed flow pattern is obtained. The basic flow is assumed to be the steady flow of two viscous, incompressible fluids at uniform depth down a plane inclined at an angle Θ with the horizontal, in a gravitational field. With the co-ordinate axes X-Y as shown in Fig. 1a with origin at the interface, the unperturbed flow is parallel to the X-axis and the velocity is a function of Y only. The upper layer is a fluid of density ρ_1 and depth d_1; and the lower layer is of density ρ_2, and depth d_2 (with $\rho_2 > \rho_1$).

The Navier-Stokes equations that govern the basic flow are

$$\left.\begin{aligned} 0 &= \rho_1 g \sin\theta + \mu \frac{d^2\bar{u}_1}{dY^2}, \qquad (1)\\ 0 &= -\frac{d\bar{p}_1}{dY} + \rho_1 g \cos\theta, \qquad (2) \end{aligned}\right\} \quad \text{for} \quad -d_1 \leq Y \leq 0,$$

$$\left.\begin{aligned} 0 &= \rho_2 g \sin\theta + \mu \frac{d^2\bar{u}_2}{dY^2}, \qquad (3)\\ 0 &= -\frac{d\bar{p}_2}{dY} + \rho_2 g \cos\theta, \qquad (4) \end{aligned}\right\} \quad \text{for} \quad 0 \leq Y \leq d_2,$$

where $\bar{u}_1$, $\bar{u}_2$ are the components of velocity of the two fluids in the X-direction, $\bar{p}_1$, $\bar{p}_2$ are the pressures, and g is the gravitational acceleration and μ is the viscosity of the two fluids, considered equal. The pressure gradient in the X-direction is zero. Since the flow is parallel to the X-axis, the equation of continuity is automatically satisfied.

Equations (1) to (4) can be integrated at once subject to the boundary conditions

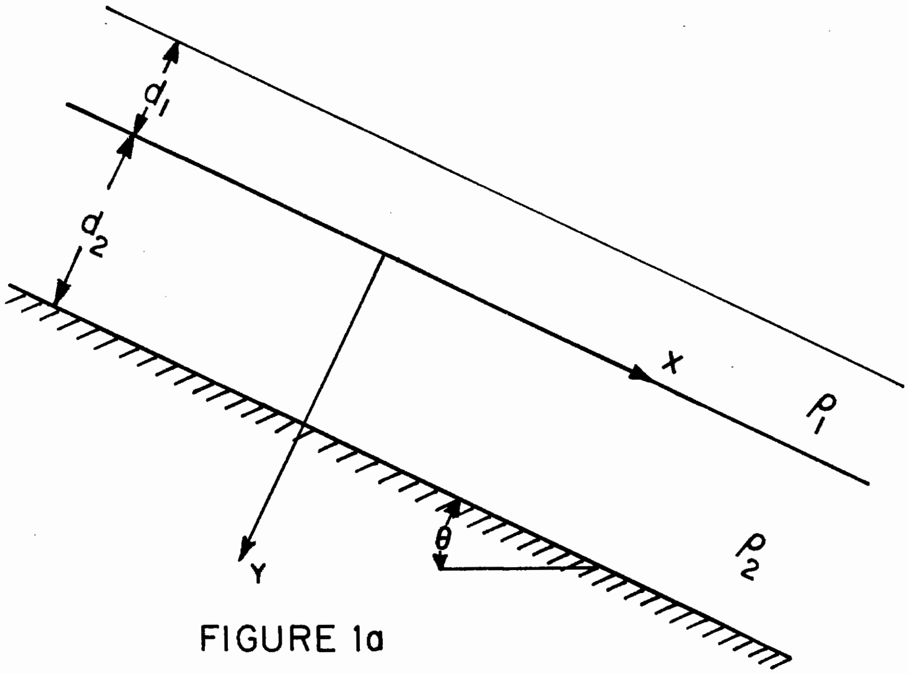

FIGURE 1a

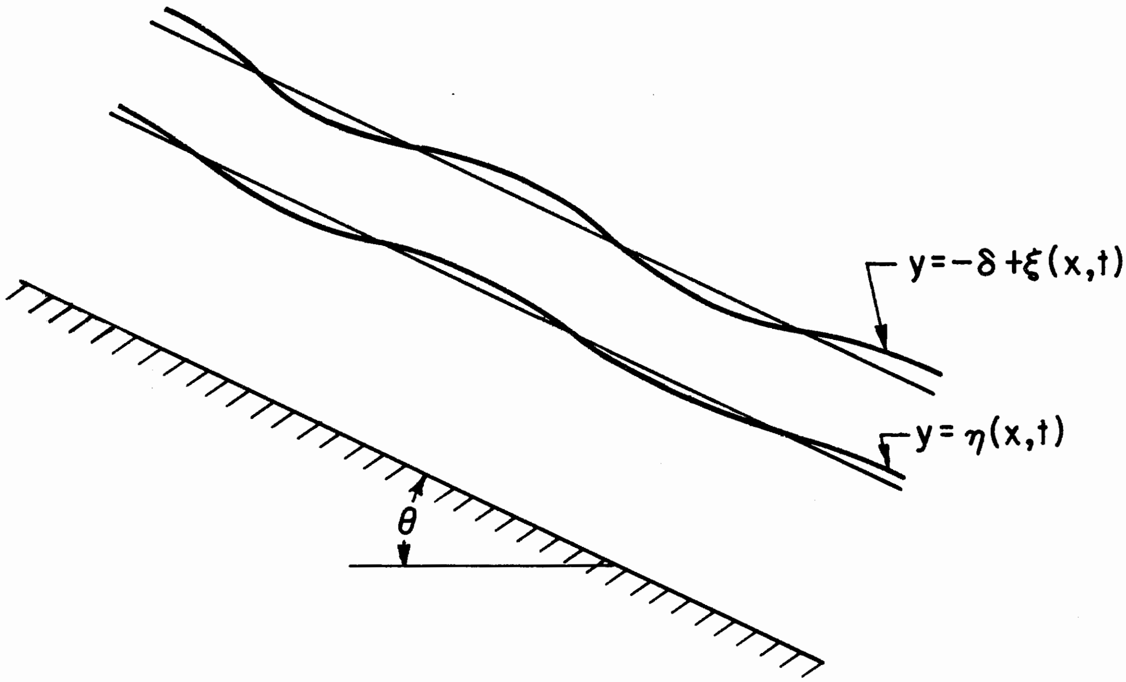

FIGURE 1b

Fig. 1. Two-layered flow down an inclined plane.

(a) Unperturbed basic flow.

(b) Disturbed flow.

$$\frac{d\bar{u}_1}{dY} = 0, \qquad \text{at } Y = d_1, \text{ (zero shear at the free-surface),}$$

$$\bar{u}_2 = 0, \qquad \text{at } Y = d_2, \text{ (no slip at the solid boundary),}$$

$$\bar{u}_2 = \bar{u}_1, \qquad \text{at } Y = 0, \text{ (no slip at the interface),}$$

and

$$\frac{d\bar{u}_2}{dY} = \frac{d\bar{u}_1}{dY}, \qquad \text{at } Y = 0, \text{ (equal shear at the interface).}$$

The solution is

$$\bar{u}_1 = \frac{\rho_1 g \sin\theta}{\mu} \left[\frac{\left(\frac{\rho_2}{\rho_1} d_2^2 - Y^2\right)}{2} + d_1 (d_2 - Y) \right], \tag{5}$$

and

$$\bar{u}_2 = \frac{\rho_2 g \sin\theta}{\mu} \left[\frac{(d_2^2 - Y^2)}{2} + \frac{\rho_2}{\rho_1} d_1 (d_2 - Y) \right]. \tag{6}$$

If we now define the average velocity $\bar{u}_a$ to be,

$$\bar{u}_a = \frac{1}{(d_1 + d_2)} \left\{ \int_{-d_1}^{0} \bar{u}_1(Y)\, dY + \int_{0}^{d_2} \bar{u}_2(Y)\, dY \right\},$$

and, after introducing the dimensionless parameters,

$$\frac{d_1}{d_2} = \delta, \qquad \frac{\rho_1}{\rho_2} = \gamma, \tag{7}$$

we have

$$\bar{u}_a = \frac{\rho_2 g \sin\theta\, d_2^2}{\mu} \left[\frac{\gamma\left(\frac{\delta}{2} + \delta^2 + \frac{\delta^3}{3}\right) + \left(\frac{1}{3} + \frac{\delta}{2}\right)}{(1 + \delta)} \right].$$

To simplify writing, we shall define the dimensionless factor

$$K \equiv \frac{1 + \delta}{\gamma\left(\frac{\delta}{2} + \delta^2 + \frac{\delta^3}{3}\right) + \left(\frac{1}{3} + \frac{\delta}{2}\right)}. \tag{8}$$

$\bar{u}_a$ can now be written as

$$\bar{u}_a = \frac{\rho_2 g \sin\theta \, d_2^2}{K\mu} . \tag{9}$$

From this expression it is natural to define the Reynolds numbers and Froude number to be

$$F = \frac{\bar{u}_a}{(g d_2)^{1/2}} , \quad R_1 = \frac{\rho_1 \bar{u}_a d_2}{\mu} , \quad R_2 = \frac{\rho_2 \bar{u}_a d_2}{\mu} \tag{10}$$

It then follows that $R_1 = \gamma R_2$, and

$$K F^2 = R_2 \sin\theta . \tag{11}$$

With $\bar{u}_a$ as the characteristic velocity and d_2 as the characteristic length, the non-dimensionalized velocities U_1 and U_2 are then

$$U_1 = K\gamma \left[\frac{(\frac{1}{\gamma} - y^2)}{2} + \delta(1-y) \right] , \tag{12}$$

and

$$U_2 = K \left[\frac{(1-y^2)}{2} + \gamma\delta(1-y) \right] . \tag{13}$$

where

$$U_1 = \frac{\bar{u}_1}{\bar{u}_a} , \quad U_2 = \frac{\bar{u}_2}{\bar{u}_a} , \quad y = \frac{Y}{d_2} .$$

Equations (12) and (13) thus give the velocity distributions in completely normalized form.

3. THE STABILITY PROBLEM

The stability problem is now formulated following the usual small perturbation technique, and with the usual procedure of considering two-dimensional disturbances only, since Squire's result (1933) and later extensions by Yih (1955) have shown that the stability or instability of a three-dimensional disturbance can be determined from that of a two-dimensional disturbance at a higher Reynolds number.

A. Equations of Motion

The Navier-Stokes equations are,

$$\frac{\partial \tilde{u}_i}{\partial \tau} + \tilde{u}_i \frac{\partial \tilde{u}_i}{\partial X} + \tilde{v}_i \frac{\partial \tilde{u}_i}{\partial Y} = -\frac{1}{\rho_i}\frac{\partial \tilde{p}_i}{\partial X} + g \sin\theta + \frac{\mu}{\rho_i}\Delta^* \tilde{u}_i ,$$

$$\frac{\partial \tilde{v}_i}{\partial \tau} + \tilde{u}_i \frac{\partial \tilde{v}_i}{\partial X} + \tilde{v}_i \frac{\partial \tilde{v}_i}{\partial Y} = -\frac{1}{\rho_i}\frac{\partial \tilde{p}_i}{\partial Y} + g \cos\theta + \frac{\mu}{\rho_i}\Delta^* \tilde{v}_i ,$$

where i = 1 denotes quantities associated with the upper fluid, and i = 2 denotes quantities associated with the lower fluid, and $\tilde{u}_i$, $\tilde{v}_i$ are the velocity components in the X, Y directions respectively, $\tilde{p}_i$ is the pressure, τ is the time, and $\Delta^* \equiv \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}$.

The continuity equation is

$$\frac{\partial \tilde{u}_i}{\partial X} + \frac{\partial \tilde{v}_i}{\partial Y} = 0 .$$

The above equations are made dimensionless by setting

$$(u_i, v_i) = \left(\frac{\tilde{u}_i}{\bar{u}_a}, \frac{\tilde{v}_i}{\bar{u}_a}\right), \qquad (x, y) = \left(\frac{X}{d_2}, \frac{Y}{d_2}\right),$$

$$p_i = \frac{\tilde{p}_i}{\rho_i \bar{u}_a^2}, \qquad t = \frac{\tau \bar{u}_a}{d_2} .$$

The nondimensional forms are then:

$$\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} + v_i \frac{\partial u_i}{\partial y} = -\frac{\partial p_i}{\partial x} + \frac{\sin\theta}{F^2} + \frac{1}{R_i}\Delta u_i , \tag{14}$$

$$\frac{\partial v_i}{\partial t} + u_i \frac{\partial v_i}{\partial x} + v_i \frac{\partial v_i}{\partial y} = -\frac{\partial p_i}{\partial y} + \frac{\cos\theta}{F^2} + \frac{1}{R_i}\Delta v_i , \tag{15}$$

$$\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y} = 0 \tag{16}$$

in which $\Delta \equiv \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

B. Perturbation Equations

Assuming small perturbations from the basic flow in the form,

$$u_i = U_i + u_i' , \quad v_i = v_i' , \quad p_i = P_i + p_i' \tag{17}$$

in which $P_i = \bar{p}_i / (\rho_i \bar{u}_a^2)$, $U_i = \bar{u}_i / \bar{u}_a$ are the dimensionless basic flow pressures and velocities, and, neglecting second order terms in the primed quantities, and making use of the fact that U_i, P_i satisfy the basic flow equations, we have, upon substitution of (17) into (14), (15) and (16), the linearized equations governing the disturbance motion,

$$\frac{\partial u_i'}{\partial t} + U_i \frac{\partial u_i'}{\partial x} + \frac{dU_i}{dy} v_i' = -\frac{\partial p_i'}{\partial x} + \frac{1}{R_i}\Delta u_i' , \tag{18}$$

$$\frac{\partial v_i'}{\partial t} + U_i \frac{\partial v_i'}{\partial x} = -\frac{\partial p_i'}{\partial y} + \frac{1}{R_i}\Delta v_i' , \tag{19}$$

$$\frac{\partial u_i'}{\partial x} + \frac{\partial v_i'}{\partial y} = 0 , \tag{20}$$

in which i = 1, or 2. From (20) it is seen at once that there exists a stream function ψ_i , such that

$$u_i' = \frac{\partial \psi_i}{\partial y} \quad , \qquad v_i' = -\frac{\partial \psi_i}{\partial x} \ .$$

We now assume a sinusoidal disturbance and write

$$\psi_i = \phi(y) \exp [i\alpha(x - ct)] \ , \tag{21}$$

and

$$p_i' = f_i(y) \exp [i\alpha(x - ct)] \ , \tag{22}$$

in which α is the dimensionless wave number defined by $2\pi d_2/\lambda$, λ being the wave length, and $c = c_r + ic_i$ is the dimensionless wave velocity. Substitution of (21) and (22) into (18) and (19) yields upon elimination of $f_i(y)$ by cross differentiation, the following two Orr-Sommerfeld equations for the two fluids,

$$\phi_1^{iv} - 2\alpha^2 \phi_1'' + \alpha^4 \phi_1 = i\alpha R_1 \{(U_1 - c)(\phi_1'' - \alpha^2 \phi_1) - U_1'' \phi_1\} \ , \tag{23}$$

in $-\delta \le y \le 0$, where the superscripts denote differentiation with respect to y, and

$$\phi_2^{iv} - 2\alpha^2 \phi_2'' + \alpha^4 \phi_2 = i\alpha R_2 \{(U_2 - c)(\phi_2'' - \alpha^2 \phi_2) - U_2'' \phi_2\} \ , \tag{24}$$

in $0 \le y \le 1$. The above two equations are now to be solved subject to eight boundary conditions, two at the free-surface, two at the solid boundary and four at the interface. The boundary conditions at the interface form the coupling between $\phi_1(y)$ and $\phi_2(y)$.

C. Boundary Conditions

Before examining the boundary conditions, we need first to study the kinematic conditions at the interface and free-surface. Let the equation of the free-surface be given by $y = -\delta + \xi(x,t)$, and the interface by $y = \eta(x,t)$. The linearized kinematic conditions are then

$$\frac{\partial \xi}{\partial t} + U_1 \frac{\partial \xi}{\partial x} = v_1' \qquad \text{, at the free-surface,}$$

and
$$\frac{\partial \eta}{\partial t} + U_2 \frac{\partial \eta}{\partial x} = v_2' = v_1' \qquad \text{, at the interface,}$$

considering ξ, and η to be of the same order as the other perturbation quantities. It then follows that

$$\xi = \frac{\phi_1(-\delta)}{c_1} \exp\left[i\alpha (x - ct) \right] , \tag{25}$$

where $c_1 \equiv c - U_1(-\delta)$,

and

$$\eta = \frac{\phi_2(0)}{c_2} \exp\left[i\alpha (x - ct) \right] , \tag{26}$$

where $c_2 \equiv c - U_2(0)$.

We now formulate the boundary conditions,† bearing in mind that the free-surface conditions are to be applied at $y = -\delta + \xi$, and the interface conditions are to be applied at $y = \eta$. However, since ξ and η are small, we need only take the leading terms, consistent with previous linearization, of the Taylor series expansions of quantities of interest and evaluate them at $y = -\delta$, or $y = 0$.

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† For a detailed derivation of the boundary conditions presented in this section, see Appendix A.

At the free surface the shear stress must vanish, and the normal stress must balance the normal stress induced by surface tension. Thus we have,

$$\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x} = 0 ,$$

$$\left(-p_1 + \frac{2}{R_1}\frac{\partial v_1}{\partial y}\right) + S_1 \frac{\partial^2 \xi}{\partial x^2} = 0 ,$$

where $S_1 \equiv T_1 / \rho_1 \bar{u}_a^2 d_2$, T_1 being the surface tension.

To the first order these equations can be written as

(i) $$\phi_1''(-\delta) + (\alpha^2 - \frac{K\gamma}{c_1})\, \phi_1(-\delta) = 0 ,$$

(ii) $$\left\{ \alpha(\gamma K \cot\theta + \alpha^2 S_1 R_1)/c_1 \right\} \phi_1(-\delta) + \alpha(R_1 c_1 + 3i\alpha)\,\phi_1'(-\delta) - i\,\phi_1'''(-\delta) = 0 .$$

At the interface, the velocity components must be continuous; hence

$$u_1 = u_2 ,$$

$$v_1 = v_2 ,$$

which, since the basic flow velocity components are equal, yield,

(iii) $$\phi_1(0) = \phi_2(0) ,$$

(iv) $$\phi_1'(0) = \phi_2'(0) .$$

The shear must also be continuous at the interface; hence

$$\left(\frac{\partial v_1}{\partial x} + \frac{\partial u_1}{\partial y}\right) = \left(\frac{\partial v_2}{\partial x} + \frac{\partial u_2}{\partial y}\right) ,$$

which to the first order is, after some calculations,

$$\text{(v)} \qquad K(1-\gamma)\phi_2(0) + c_2\phi_1''(0) - c_2\phi_2''(0) = 0 .$$

The difference of the normal stresses must be balanced by the normal stress induced by surface tension at the interface; hence

$$\left(-p_2 + \frac{2}{R_2}\frac{\partial v_2}{\partial y}\right) - \left(-p_1 + \frac{2}{R_1}\frac{\partial v_1}{\partial y}\right)\gamma + S_2\frac{\partial^2 \eta}{\partial x^2} = 0$$

where $S_2 \equiv T_2 / \rho_2 \bar{u}_a^2 d_2$, T_2 being the interfacial surface tension. Again, after some calculations, to the first order, we have,

$$\text{(vi)} \qquad c_2\left(\phi_2'''(0) - \phi_1'''(0)\right) + (1-\gamma)\, i\alpha R_2 c_2 \left[c_2\phi_1'(0) - K\gamma\delta\,\phi_1(0)\right] + $$
$$+\, i\alpha\left[K(1-\gamma)\cot\theta + \alpha^2 S_2 R_2\right]\phi_1(0) = 0 .$$

At the solid boundary, y = 1, we have $u' = 0$, $v' = 0$. Thus

$$\text{(vii)} \qquad \phi_2(1) = 0 ,$$

$$\text{(viii)} \qquad \phi_2'(1) = 0 .$$

D. Eigenvalue Problem

Equation (23) and (24) together with boundary conditions (i) to (viii) is the eigenvalue problem we wish to solve with c as the eigenvalue. The general solutions of (23) and (24) will contain eight arbitrary constants. The substitution of these solutions into the eight homogeneous boundary conditions will yield eight homogeneous algebraic equations for the eight constants. The vanishing of the determinant of the coefficients will then give the secular equation of the form

$$F(\alpha, R_1, \gamma, \delta, \theta, c) = 0,$$

or

$$c = c(\alpha, R_1, \gamma, \delta, \theta),$$

from which the eigenvalue is determined with $c_i > 0$ representing growing disturbances, and $c_i < 0$ representing damped disturbances, and $c_i = 0$ representing neutral oscillations. Since this relationship is complex, it can be resolved into the relationships

$$c_r = c_r(\alpha, R_1, \gamma, \delta, \theta),$$

and

$$c_i = c_i(\alpha, R_1, \gamma, \delta, \theta).$$

Setting $c_i = 0$, we obtain a relationship between α (wave number) and R_1 (Reynolds number) for given values of γ (density ratio), δ (depth ratio) and θ (slope angle). This relationship between α and R_1 represents a curve in the α-R_1 plane, which is the curve of neutral stability. We note further that the special case $\gamma = 1$, $\delta = 0$, $S_2 = 0$, corresponds to a one-layered homogeneous system, which has been treated by Benjamin (1957) and Yih (1963). In subsequent calculations this limiting case will be calculated and the results checked with those obtained by Benjamin and Yih.

4. SOLUTION OF THE EIGENVALUE PROBLEM

Direct solution by series method is very lengthy. However useful information can be obtained by examining suitable asymptotic limits. In particular we shall seek asymptotic solutions for two cases: (A). case for long waves (small α); and (B). case for short waves (large α). It will be seen that most of the relevant information that we desire can be extracted from these two cases.

A. Case for Long Waves (Small α)

The stability of the system with respect to long waves will be examined both with respect to surface waves and shear waves. Yih's (1963) perturbation procedure, which leads to a study of surface waves, will be used. It is to be noted that this is a "regular" perturbation procedure and does not introduce any difficulty usually encountered in the study of hydrodynamic stability problems for high Reynolds number where the asymptotic solutions are obtained by a "singular" perturbation procedure.

We introduce perturbation series of the form

$$\phi_1 = \phi_{10} + \phi_{11} + \phi_{12} + \ldots \tag{27}$$

$$\phi_2 = \phi_{20} + \phi_{21} + \phi_{22} + \ldots \tag{28}$$

$$c = c_0 + \Delta c + \Delta^2 c + \ldots \tag{29}$$

where

$$\phi_{10},\ \phi_{20},\ c_0 \sim O(\alpha^0)\ ; \qquad \phi_{11},\ \phi_{21},\ \Delta c \sim O(\alpha^1)\ ;$$

$$\phi_{12},\ \phi_{22},\ \Delta^2 c \sim O(\alpha^2)\ ; \qquad \text{etc.}$$

(a) Zeroth Order Solution. Substitution of (27), (28), and (29) into (23) and (24) and (i) to (viii) and collecting terms of order α°, yields,

$$\phi_{10}^{IV} = 0 \quad , \qquad -\delta \le y \le 0 , \tag{30}$$

$$\phi_{20}^{IV} = 0 \quad , \qquad 0 \le y \le 1 , \tag{31}$$

(i) $\phi_{10}''(-\delta) - \dfrac{K\gamma}{c_{10}} \phi_{10}(-\delta) = 0$,

(ii) $\phi_{10}'''(-\delta) = 0$,

(iii) $\phi_{10}(0) - \phi_{20}(0) = 0$,

(iv) $\phi_{10}'(0) - \phi_{20}'(0) = 0$,

(v) $K(1-\gamma)\phi_{20}(0) + c_{20}\phi_1''(0) - c_{20}\phi_{20}''(0) = 0$,

(vi) $\phi_{20}'''(0) - \phi_{10}'''(0) = 0$,

(vii) $\phi_{20}(1) = 0$,

(viii) $\phi_{20}'(1) = 0$.

where $c_{10} = c_o - U_1(-\delta)$ and $c_{20} - U_2(o)$. It is to be noted that in this reduced zeroth order eigenvalue problem, the eigenvalue no longer appears in the differential equation so that no information can be obtained regarding shear waves. The shear waves will be examined separately in a separate calculation below in subsection (c).

The solution of this eigenvalue problem is straightforward[†]. After some calculations, we find that the wave velocity, c_o, is

$$c_o = \frac{K}{4}(3 + 6\gamma\delta + 2\gamma\delta^2) \pm \sqrt{(1/4 + \gamma^2\delta^4 + 2\gamma^2\delta^3 + 3\gamma^2\delta^2 + \gamma\delta - \gamma\delta^2)}, \tag{32}$$

The corresponding eigenfunctions determined up to a multiplicative constant, which can be chosen to be unity without loss of generality, are

$$\phi_{10} = \left\{ 1 - 2y + \frac{\gamma(1+2\delta)y^2}{\frac{1}{2}(1+2\gamma\delta-2\gamma\delta^2) \pm \sqrt{(1/4 + \gamma^2\delta^4 + 2\gamma^2\delta^3 + 3\gamma^2\delta^2 + \gamma\delta - \gamma\delta^2)}} \right\} \tag{33}$$

and

$$\phi_{20} = (1-y)^2 . \tag{34}$$

We note that the quantity under the radical is positive definite for any value of δ. This can be shown as follows: for $0 < \delta \leq 1$, this is obvious. For $\delta > 1$, the quantity can be written as

$$1/4 - \left[\gamma(\delta^2 - \delta) - \gamma^2(\delta^4 + 2\delta^3 + 3\delta^2)\right].$$

The expression in the square bracket attains a maximum at

$$\gamma = \frac{\delta^2 - \delta}{2(\delta^4 + 2\delta^3 + 3\delta^2)},$$

and equals

$$\frac{(\delta^2 - \delta)^2}{4(\delta^4 + 2\delta^3 + 3\delta^2)},$$

† See Appendix B for a detailed derivation.

which is always less than 1/4 for $\delta > 1$. This concludes the demonstration.

It then follows that c_o is real for wave number $\alpha = 0$, which means that the line $\alpha = 0$ in the α - R_1 plane is part of the neutral curve whatever the value of γ, δ, θ, and R_1. This is a very useful and welcome piece of information, for it shows that neutral oscillations can exist right down to Reynolds number $R_1 = 0$.

So far we have not yet discussed the sign in front of the radical for the eigenvalue in equation (32). It appears that the plus and minus signs correspond to two different modes of waves. If both of these eigenvalues were admissible for our calculation of the neutral curve, then there would be two neutral curves in the α - R_1 plane, one corresponding to each mode in contradiction to the general problem set forth in Section 3. One of the modes is thus inadmissible for such calculation.

We observe that for the special case of homogeneous fluid ($\gamma = 1$, $\delta = 0$), we recover the results of Benjamin and Yih, i.e., $c_o = 3$, $\phi_{10} = \phi_{20} = (1-y)^2$, only when the positive sign is taken in front of the radical. Hence the positive sign is the one that is to be used for subsequent calculations.

The ratio of the amplitudes of the free-surface and interface, r, is given by

$$r = \frac{\phi_{10}(-\delta)/c_{10}}{\phi_{20}(0)/c_{20}} ,$$

where

$$c_{10} = \frac{K}{4}(1+2\gamma\delta) + \frac{K}{2}\sqrt{(1/4 + \gamma^2\delta^4 + 2\gamma^2\delta^3 + 3\gamma^2\delta^2 + \gamma\delta - \gamma\delta^2)} ,$$

$$c_{20} = \frac{K}{4}(1+2\gamma\delta + 2\gamma\delta^2) + \frac{K}{2}\sqrt{(1/4 + \gamma^2\delta^4 + 2\gamma^2\delta^3 + 3\gamma^2\delta^2 + \gamma\delta - \gamma\delta^2)} .$$

Therefore

$$r = \left[1 + 2\delta + \frac{\gamma(1+2\delta)\,\delta^2}{\frac{1}{2}(1+2\gamma\delta-2\gamma\delta^2)+\sqrt{(\frac{1}{4}+\gamma^2\delta^4+2\gamma^2\delta^3+3\gamma^2\delta^2+\gamma\delta-\gamma\delta^2)}} \right] \frac{c_{20}}{c_{10}} \tag{35}$$

It is easily seen that r is positive definite for $\gamma < 1$ and all δ; since $\sqrt{(\frac{1}{4}+\gamma^2\delta^4+2\gamma^2\delta^3+3\gamma^2\delta^2+\gamma\delta-\gamma\delta^2)}$ is positive definite as shown earlier, and $\frac{1}{2}(1+2\gamma\delta-2\gamma\delta^2)+\sqrt{(\frac{1}{4}+\gamma^2\delta^4+2\gamma^2\delta^3+3\gamma^2\delta^2+\gamma\delta-\gamma\delta^2)}$ is never equal to zero. It may be of interest to record here that when the negative sign is used in front of the radical, calculations have shown that r would be negative, indicating an oscillation 180° out of phase.

<u>(b) First Order Solution.</u> The first order approximation is obtained by collecting terms of order α^1, which yields the following nonhomogeneous differential system. The Orr-Sommerfeld equation now becomes,

$$\phi_{11}^{IV} = i\alpha R_1 \left\{ (U_1 - c_0)\,\phi_{10}'' - U_1''\,\phi_{10} \right\}, \quad -\delta \le y \le 0, \tag{36}$$

$$\phi_{21}^{IV} = i\alpha R_2 \left\{ (U_2 - c_0)\,\phi_{20}'' - U_2''\,\phi_{20} \right\}, \quad 0 \le y \le 1. \tag{37}$$

In these equations the right hand sides of course are known.

The boundary conditions are now

(i) $$\phi_{11}''(-\delta) - \frac{K\gamma}{c_{10}}\,\phi_{11}(-\delta) = -\frac{K\gamma\,\Delta c}{(c_{10})^2}\,\phi_{10}(-\delta),$$

(ii) $$\phi_{11}'''(-\delta) = -i\alpha\left\{ \left(\frac{\gamma K \cot\theta + \alpha^2 S_1 R_1}{c_{10}} \right)\phi_{10}(-\delta) + R_1 c_{10}\,\phi_{10}'(-\delta) \right\},$$

(iii) $\phi_{11}(0) - \phi_{21}(0) = 0$,

(iv) $\phi_{11}'(0) - \phi_{21}'(0) = 0$,

(v) $K(1-\gamma)\,\phi_{21}(0) + c_{20}\,[\phi_{11}''(0) - \phi_{21}''(0)] = \Delta c\,[\phi_{20}''(0) - \phi_{10}''(0)]$,

(vi) $\phi_{21}'''(0) - \phi_{11}'''(0) = i\alpha\,[-(1-\gamma)R_2\{c_{20}\,\phi_{10}'(0) - K\gamma\delta\,\phi_{10}(0)\} -$
$- \left\{\dfrac{K(1-\gamma)\cot\theta + \alpha^2 S_2 R_2}{c_{20}}\right\}\phi_{10}(0)\,]$,

(vii) $\phi_{21}(1) = 0$,

(viii) $\phi_{21}'(1) = 0$.

It will now be noted that the left hand side of this system has the same form as the left hand side of the zeroth order system as it should be from the theory of regular perturbation analysis. The general solutions ϕ_{11} and ϕ_{21} can again be determined at once by direct integration, and there will again appear eight arbitrary constants. Substitution of ϕ_{11} and ϕ_{21} into the boundary conditions yields eight linear non-homogeneous algebraic equations with the eight constants as unknowns. The determinant of the coefficient is now known to be zero, since they are the same as the zeroth order calculation with c_0 assuming the value determined previously. Thus Δc can be calculated.

The calculations involved are very lengthy[†]. The final result is

$$\Delta c = i\alpha\left\{\frac{G}{H}R_1 - \left(\frac{\Phi}{H}\cot\theta + \frac{\Lambda}{H}\alpha^2\right)\right\} , \qquad (38)$$

[†] For details of the calculations, see Appendix C.

where

$$G = \Big\{ 2c_{20}c_{10}\,[\,(6c_{10}\delta - K\gamma\delta^3)(-\frac{\delta^2 A}{12} - \frac{c_{10}\phi'_{10}(-\delta)}{6}) - (K\gamma\frac{A\delta^5}{30} - \frac{c_{10}A\delta^3}{3})\,] -$$
$$- [\,2c_{20}c_{10}K\gamma + c_{10}K(1-\gamma)(2c_{10} - K\gamma\delta^2)\,]\cdot[\,\frac{1}{\gamma}(\frac{K(\gamma\delta+1)}{15} + \frac{A\delta}{8}) + \frac{A\delta^2}{6} +$$
$$+ \frac{1}{3}c_{10}\phi'_{10}(-\delta) + \frac{(1-\gamma)}{3\gamma}(c_{20}\phi'_{10}(0) - K\gamma\delta)\,] - 2c_{20}c_{10}K\gamma\delta\,[\,\frac{1}{\gamma}(\frac{K(\gamma\delta+1)}{12} +$$
$$+ \frac{A\delta}{6}) + \frac{A\delta^2}{4} + \frac{1}{2}c_{10}\phi'_{10}(-\delta) + \frac{(1-\gamma)}{2\gamma}(c_{20}\phi'_{10}(0) - K\gamma\delta)\,]\Big\}\,,$$

$$\Phi = \tfrac{1}{3}c_{20}(6c_{10}\delta - K\gamma\delta^3)K\gamma\,\phi_{10}(-\delta) + [\,\tfrac{1}{3}(2c_{20}K\gamma + K(1-\gamma)(2c_{10} - K\gamma\delta^2)) +$$
$$+ K\gamma\delta c_{20}\,]\cdot[\,K\gamma\,\phi_{10}(-\delta) + \frac{c_{10}}{c_{20}}K(1-\gamma)\,]\,,$$

$$H = c_{10}(2c_{10} - K\gamma\delta^2)(\phi''_{20}(0) - \phi''_{10}(0)) + 2c_{20}K\gamma\,\phi_{10}(-\delta)\,,$$

$$A = 2K\gamma + \frac{2K\gamma^2\delta(1+2\gamma\delta)}{\tfrac{1}{2}(1+2\gamma\delta - 2\gamma\delta^2) + \sqrt{(\tfrac{1}{4} + \gamma^2\delta^4 + 2\gamma^2\delta^3 + 3\gamma^2\delta^2 + \gamma\delta - \gamma\delta^2)}}\,,$$

$$\lambda = \Big\{ \tfrac{1}{3}c_{20}(6c_{10}\delta - K\gamma\delta^3)K\gamma\,\phi_{10}(-\delta)S_1R_1 + [\,\tfrac{1}{3}(2c_{20}c_{10}K\gamma +$$
$$+ c_{10}K(1-\gamma)(2c_{20} - K\gamma\delta^2)) + K\gamma\delta c_{20}c_{10}\,]\cdot[\,\frac{K\gamma\,\phi_{10}(-\delta)}{c_{10}}S_1R_1 +$$
$$+ \frac{K(1-\gamma)}{c_{20}}S_2R_2\,]\Big\}\,.$$

Since G, Φ, H, λ, and A are all real for given values of γ and δ, it then follows that Δc is purely imaginary. Moreover numerical computations indicate that G, Φ, and H are all positive. Thus $\Delta c = ic_i$, and c_i will increase or decrease from zero when α increases from zero, according as

$$R > \frac{\Phi}{G}\cot\theta \qquad \text{or} \qquad R < \frac{\Phi}{G}\cot\theta\,. \tag{39}$$

Hence, the neutral stability curve has a bifurcation point on $\alpha = 0$, at $R = (\Phi/G) \cot\theta$.

For the special case when $\gamma = 1$, $\delta = 0$, and $S_2 = 0$, we have $R_1 = R_2 = R$, and

$$c_i = \frac{6\alpha R}{5} - \frac{\alpha(3\cot\theta + \alpha^2 S R)}{3}$$

recovering the result given by Yih (1963).

The numerical results obtained for the two-layered system will be discussed in detail in Section 5 below.

(c) Shear Waves. In order to complete the stability study for long waves, we must next investigate the shear waves, which, as noted at the beginning of this section have been dropped out of the calculations. In order to include these waves, we must now assume that although α is small αc is not small. The Orr-Sommerfeld equations then become

$$\phi_1^{iv} + i\alpha R_1 c\, \phi_1'' = 0, \tag{40}$$

$$\phi_2^{iv} + i\frac{\alpha R_1}{\gamma} c\, \phi_2'' = 0, \tag{41}$$

and the boundary conditions are now

(i) $\phi_1''(-\delta) = 0,$

(ii) $\beta^2 \phi_1'(-\delta) - \phi_1'''(-\delta) = 0,$

(iii) $\phi_1(0) - \phi_2(0) = 0,$

(iv) $\phi_1'(0) - \phi_2'(0) = 0,$

(v) $\phi_1''(0) - \phi_2''(0) = 0$,

(vi) $\phi_2'''(0) - \phi_1'''(0) - \left(\frac{1-\gamma}{\gamma}\right)\beta^2 \phi_1'(0) = 0$,

(vii) $\phi_2(1) = 0$,

(viii) $\phi_2'(1) = 0$,

where we have written β^2 for $-i\alpha R_1 c$. This again is a homogeneous differential system with c as the eigenvalue. The general solutions of (40) and (41) are

$$\phi_1 = A_1 + B_1 y + C_1 e^{\beta y} + D_1 e^{-\beta y}, \tag{42}$$

$$\phi_2 = A_2 + B_2 y + C_2 e^{\frac{\beta}{\sqrt{\gamma}} y} + D_2 e^{-\frac{\beta}{\sqrt{\gamma}} y}, \tag{43}$$

where A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2, are eight arbitrary constants. Substitution of (42) and (43) into the eight homogeneous boundary conditions (i) through (viii) once more yields a system of eight homogeneous linear algebraic equations to determine the eight arbitrary constants. In order to have nontrivial solutions, the determinant of the coefficients must vanish, which gives the secular equation to determine c. After some straightforward calculations,[†] we obtain the following secular equation governing c:

$$\cosh\frac{\beta}{\sqrt{\gamma}} \cosh \beta\delta + \sqrt{\gamma} \sinh\frac{\beta}{\sqrt{\gamma}} \sinh \beta\delta = 0. \tag{44}$$

For the limiting case $\gamma = 1$, $\delta = 0$, equation (44) becomes

$$\cosh \beta = 0,$$

[†]See Appendix D for a detailed derivation of the secular equation.

recovering the result for a one-layered homogeneous fluid.† Since c is in general complex β is complex. Separating into real and imaginary parts $\beta = \beta_r + i\beta_i$ we have, from (44), that

$$\cosh\frac{\beta_r}{\sqrt{\gamma}}\cosh\delta\beta_r\left(\frac{1}{\sqrt{\gamma}}\cos\frac{\beta_i}{\sqrt{\gamma}}\cos\delta\beta_i - \sin\frac{\beta_i}{\sqrt{\gamma}}\sin\delta\beta_i\right) -$$
$$- \sinh\frac{\beta_r}{\sqrt{\gamma}}\sinh\delta\beta_r\left(\frac{1}{\sqrt{\gamma}}\sin\frac{\beta_i}{\sqrt{\gamma}}\sin\delta\beta_i - \cos\frac{\beta_i}{\sqrt{\gamma}}\cos\delta\beta_i\right) = 0\,,$$

$$\sinh\frac{\beta_r}{\sqrt{\gamma}}\cosh\delta\beta_r\left(\frac{1}{\sqrt{\gamma}}\sin\frac{\beta_i}{\sqrt{\gamma}}\cos\delta\beta_i + \sin\delta\beta_i\cos\frac{\beta_i}{\sqrt{\gamma}}\right) +$$
$$+ \sinh\delta\beta_r\cosh\frac{\beta_r}{\sqrt{\gamma}}\left(\frac{1}{\sqrt{\gamma}}\sin\delta\beta_i\cos\frac{\beta_i}{\sqrt{\gamma}} + \sin\frac{\beta_i}{\sqrt{\gamma}}\cos\delta\beta_i\right) = 0\,.$$

The roots are then given by $\beta_r = 0$, and β_i satisfying

$$\frac{1}{\sqrt{\gamma}}\cos\frac{\beta_i}{\sqrt{\gamma}}\cos\delta\beta_i - \sin\frac{\beta_i}{\sqrt{\gamma}}\sin\delta\beta_i = 0\,,$$

or

$$\tan\frac{\beta_i}{\sqrt{\gamma}}\tan\delta\beta_i = \frac{1}{\sqrt{\gamma}}\,. \qquad (45)$$

There is a denumerably infinite number of real roots for (45), all non-zero. Thus β is purely imaginary. Hence β^2 is always a negative number, say $-M^2$, where M^2 is positive. Hence, $\alpha R_1(c_r + ic_i) = i\beta^2$, or $\alpha R_1 c_i = -M^2$, showing the damped nature of the shear waves.

It is now safe to conclude that the stability for long waves is indeed governed by surface waves.

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† Yih's result for this case contains a minor algebraic error. His conclusions however are unaffected by this error.

B. Case for Short Waves (Large α)

For any finite Reynolds number, and for α very large, and provided c is small compared with α, (more precisely of order α^{-2}), the asymptotic form of the Orr-Sommerfeld equations can be written as

$$\phi_1^{iv} - 2\alpha^2\phi_1'' + \alpha^4\phi_1 = 0 , \qquad -\delta \leq y \leq 0 , \tag{46}$$

$$\phi_2^{iv} - 2\alpha^2\phi_2'' + \alpha^4\phi_2 = 0 , \qquad 0 \leq y \leq 1 . \tag{47}$$

with the boundary conditions

(i) $\phi_1''(-\delta) + (\alpha^2 - \frac{K\gamma}{c_1})\phi_1(-\delta) = 0$,

(ii) $-\{i\alpha(\gamma K \cot\theta + \alpha^2 S_1 R_1)/c_1\}\phi_1(-\delta) + 3\alpha^2\phi_1'(-\delta) - \phi_1'''(-\delta) = 0$,

(iii) $\phi_1(0) - \phi_2(0) = 0$,

(iv) $\phi_1'(0) - \phi_2'(0) = 0$,

(v) $K(1-\gamma)\phi_2(0) + c_2\phi_1''(0) - c_2\phi_2''(0) = 0$,

(vi) $c_2[\phi_2'''(0) - \phi_1'''(0)] + i\alpha[K(1-\gamma)\cot\theta + \alpha^2 S_2 R_2]\phi_1(0) = 0$,

(vii) $\phi_2(1) = 0$,

(viii) $\phi_2'(1) = 0$.

The above eigenvalue problem, with c as the eigenvalue, is true even for the Reynolds number approaching or equal to zero. Since as $R_1 \to 0$, $R_2 \to 0$ and $\bar{u}_a \to 0$ for finite μ. But

$$\mu\bar{u}_a = \frac{\rho_2 g \sin\theta\, d_2^2}{K}$$ is finite. Hence

$S_1R_1 = {}^{T}1/\mu\bar{u}_a$, and $S_2R_2 = {}^{T}2/\mu\bar{u}_a$ are finite quantities even for R_1 and R_2 approaching zero or in the limit equal to zero.

The solutions for (46) and (47) are,

$$\phi_1 = A_1 e^{\alpha y} + B_1 e^{-\alpha y} + C_1 y e^{\alpha y} + D_1 y e^{-\alpha y}, \tag{48}$$

and

$$\phi_2 = A_2 e^{\alpha y} + B_2 e^{-\alpha y} + C_2 y e^{\alpha y} + D_2 y e^{-\alpha y}. \tag{49}$$

From the boundary conditions, we once more obtain a secular equation by setting the determinant of the coefficients of A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2 to zero. After some straightforward substitution we obtain the following 8x8 determinantal equation:

$$\begin{vmatrix}
\left(2\alpha^2-\frac{K\gamma}{c_1}\right)e^{-\alpha\delta} & \left(2\alpha^2-\frac{K\gamma}{c_1}\right)e^{\alpha\delta} & \left[2\alpha-\left(2\alpha^2-\frac{K\gamma}{c_1}\right)\delta\right]e^{-\alpha\delta} & \left[-2\alpha-\left(2\alpha^2-\frac{K\gamma}{c_1}\right)\delta\right]e^{\alpha\delta} & 0 & 0 & 0 & 0 \\
\left(-\frac{i\alpha M}{c_1}+2\alpha^3\right)e^{-\alpha\delta} & \left(-\frac{i\alpha M}{c_1}-2\alpha^3\right)e^{\alpha\delta} & \left(\frac{i\alpha M}{c_1}-2\alpha^3\right)\delta e^{-\alpha\delta} & \left(\frac{i\alpha M}{c_1}+2\alpha^3\right)\delta e^{\alpha\delta} & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \\
\alpha & -\alpha & 1 & 1 & -\alpha & \alpha & -1 & -1 \\
\alpha^2 c_2 & \alpha^2 c_2 & 2\alpha c_2 & -2\alpha c_2 & [K(1-\gamma)-\alpha^2 c_2] & [K(1-\gamma)-\alpha^2 c_2] & -2\alpha c_2 & 2\alpha c_2 \\
(i\alpha N-\alpha^3 c_2) & (i\alpha N+\alpha^3 c_2) & -3\alpha^2 c_2 & -3\alpha^2 c_2 & \alpha^3 c_2 & -\alpha^3 c_2 & 3\alpha^2 c_2 & 3\alpha^2 c_2 \\
0 & 0 & 0 & 0 & e^{\alpha} & e^{-\alpha} & e^{\alpha} & e^{-\alpha} \\
0 & 0 & 0 & 0 & \alpha e^{\alpha} & -\alpha e^{-\alpha} & (1+\alpha)e^{\alpha} & (1-\alpha)e^{-\alpha}
\end{vmatrix} = 0, \tag{50}$$

with M written for $(\gamma K \cot\theta + \alpha^2 S_1 R_1)$ and N for $[K(1-\gamma)\cot\theta + \alpha^2 S_2 R_2]$, and

$$K = \frac{1+\delta}{\gamma\left(\frac{\delta}{2} + \delta^2 + \frac{\delta^3}{3}\right) + \left(\frac{1}{3} + \frac{\delta}{2}\right)} ,$$

$$c_1 = c - \frac{K}{2}(1 + 2\gamma\delta + \gamma\delta^2) ,$$

and $c_2 = c - \frac{K}{2}(1 + 2\gamma\delta)$, as before.

The expansion of the determinant will then yield an algebraic equation to determine c. However this process is very laborious, and since our interest is in the value of c when α is large, we need only take a look at the roots of c as $\alpha \to \infty$. The determinant, on expansion† and taking the limit as $\alpha \to \infty$, gives

$$2\alpha\delta \cosh\alpha\delta \sinh^2\alpha\delta \,(\sinh\alpha\delta + \cosh\alpha\delta)(i\alpha^3 S_1 R_1 + 2\alpha^2 c_1) \cdot (i\alpha^3 S_2 R_2 + 2\alpha^2 c_2) = 0 , \tag{51}$$

Hence as $\alpha \to \infty$

$$c \to \frac{K}{2}(1 + 2\gamma\delta + \gamma\delta^2) - \frac{1}{2} i S_1 R_1 ,$$

or $$c \to \frac{K}{2}(1 + 2\gamma\delta) - \frac{1}{2} i S_2 R_2 .$$

---

†The full expansion of the determinant was taken. See Appendix E for details.

Now since S_1 and S_2 are positive for nonzero surface tension, therefore very short waves are damped by surface tension. On the other hand since S_1R_1 and S_2R_2 vary inversely as μ, viscosity reduces the rate of damping, a fact pointing to the dual role of viscosity noted by Yih (1963).

From the above discussion for the asymptotic solutions for long waves and short waves, the general trend of the neutral stability curve is determined. A typical sketch of a neutral stability curve is shown in Figure 2. Detailed calculations for part of the neutral curve for long waves are discussed in the next section.

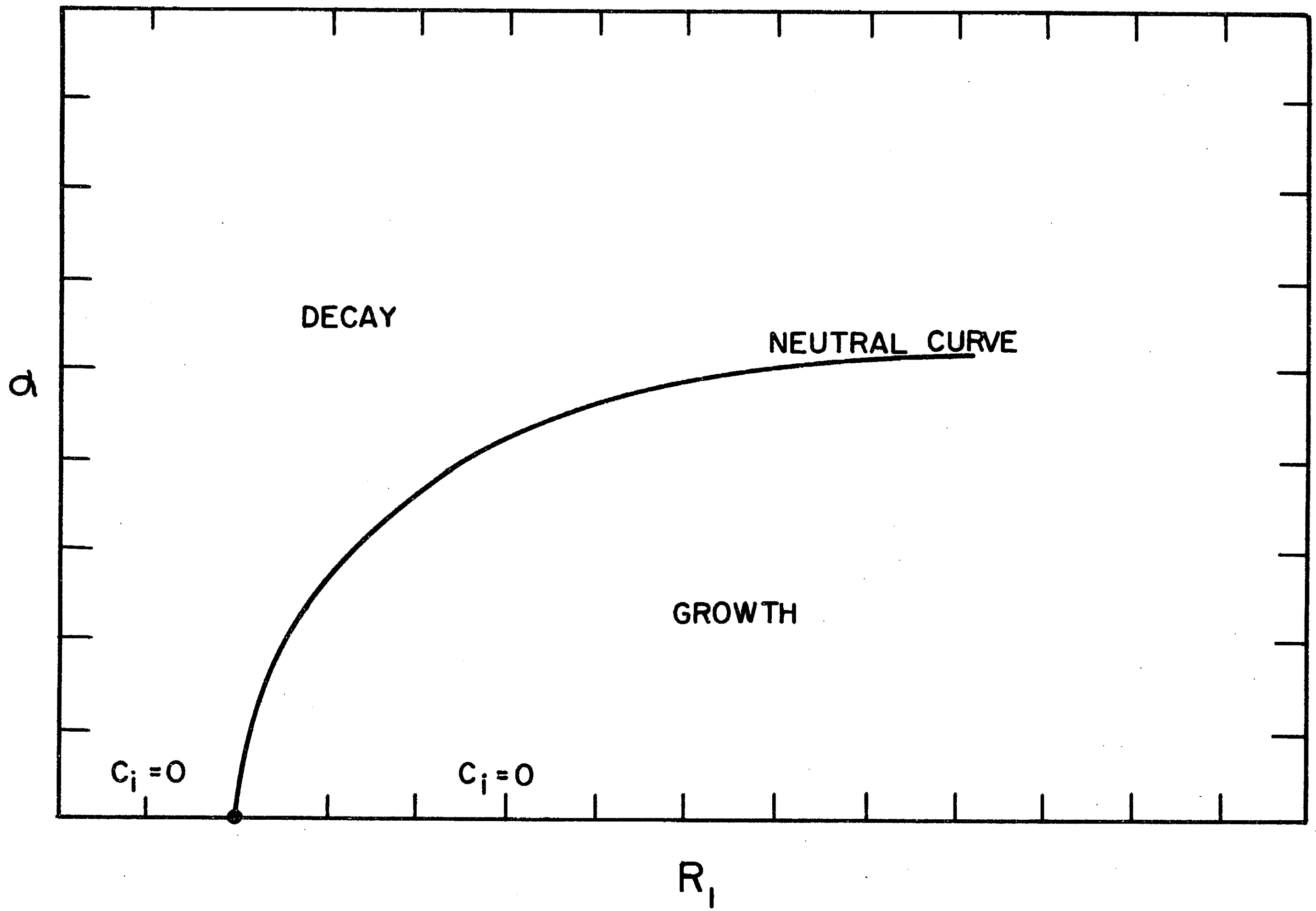

FIGURE 2

Fig. 2. Free sketch of neutral curve.

5. DISCUSSIONS OF GRAPHS

From the algebraic results obtained in the previous section for long waves, numerical results are easily computed and presented in the form of graphs as shown in Figures 3 through 7. In this section we shall discuss these graphs.

Figure 3 shows the variation of wave speed c_o with depth ratio δ for different values of the density ratio γ for the limiting case of long waves. For small density differences ($\gamma \approx 1$), the wave speed remains essentially constant and equal to 3, the wave speed for the homogeneous one-layered flow. For larger and larger density differences (decreasing γ) the wave speed is reduced more and more; the greatest reduction occurring when the depth of the upper fluid is from one to two times larger than the depth of the lower fluid. As the depth of the upper fluid becomes much larger than the depth of the lower fluid ($\delta >> 1$), the wave speed goes asymptotically to 3 as is to be expected since the disturbance is mainly associated with the free surface.

Figure 4 shows that for long waves the ratio of the amplitudes of the free surface and interface, r, increases as the depth ratio $\delta = d_1/d_2$ increases, and is quite insensitive to the ratio of densities. This is to be expected since the two surface oscillations are in phase with each other and the mode represents mainly a free-surface mode.

Figure 5 gives the critical Reynolds number as a function of the depth ratio and density ratio based on computed bifurcation point of the neutral curve on $\alpha = 0$. The figure indicates that the presence of the upper layer destabilizes the flow as compared with that of a single layer in that it shifts the bifurcation point on $\alpha = 0$ to a lower Reynolds number for a constant angle of inclination of the plane. For an upper layer which is shallower than the lower layer, a smaller density ratio between the upper and lower fluid tends to make the flow more unstable. When the depth of the upper layer exceeds that of the lower layer by three times, then the stability characteristic is insensitive to the ratio of the densities.

Figure 6 shows a typical plot of curves of constant c_i for small wave number (long waves) and small Reynolds numbers. They exhibit the expected behavior: c_i increases for larger values of the wave number and Reynolds number. For long waves the stabilizing influence of surface tension on the curves of constant growth rate is small, and does not effect these curves appreciably.

Figure 7 shows the effect of surface tension on the neutral curve for small wave numbers and small Reynolds numbers. It is assumed for convenience that the surface tension parameters S_1R_1 and S_2R_2 are equal. It can be seen that surface tension has a stabilizing effect for this range of α and R_1, and reduces the range of α for which instability occurs for any constant R_1.

6. SUMMARY OF CONCLUSIONS

The relevant results of this study will now be summarized.

(a) The axis $\alpha = 0$ in the α - R_1 plane is part of the neutral stability curve, showing that neutral oscillation can exist right down to $R_1 = 0$.

(b) There exists a bifurcation point of the neutral stability curve on $\alpha = 0$, which marks the critical Reynolds number above which there are unstable disturbances.

(c) The addition of an upper layer destabilizes a one-layer free-surface flow.

(d) When the depth of the upper layer is large compared with the lower layer, the bifurcation point is insensitive to the ratio of densities.

(e) For long waves, the stabilizing influence of surface tension on curves of constant growth rate is small.

(f) Surface tension has a stabilizing effect for long waves and small Reynolds number.

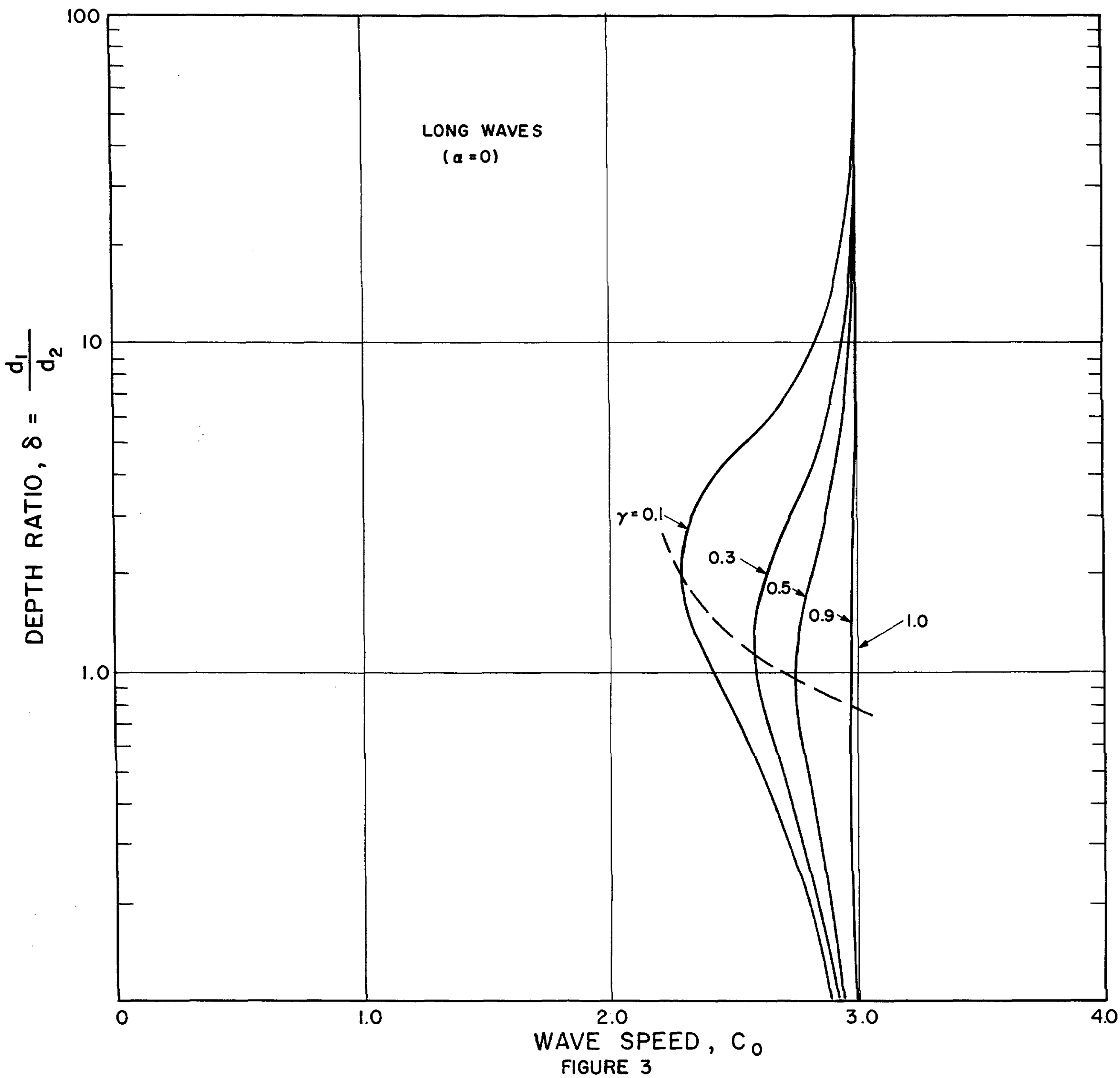

Fig. 3. Wave speed of long waves for constant density ratios.

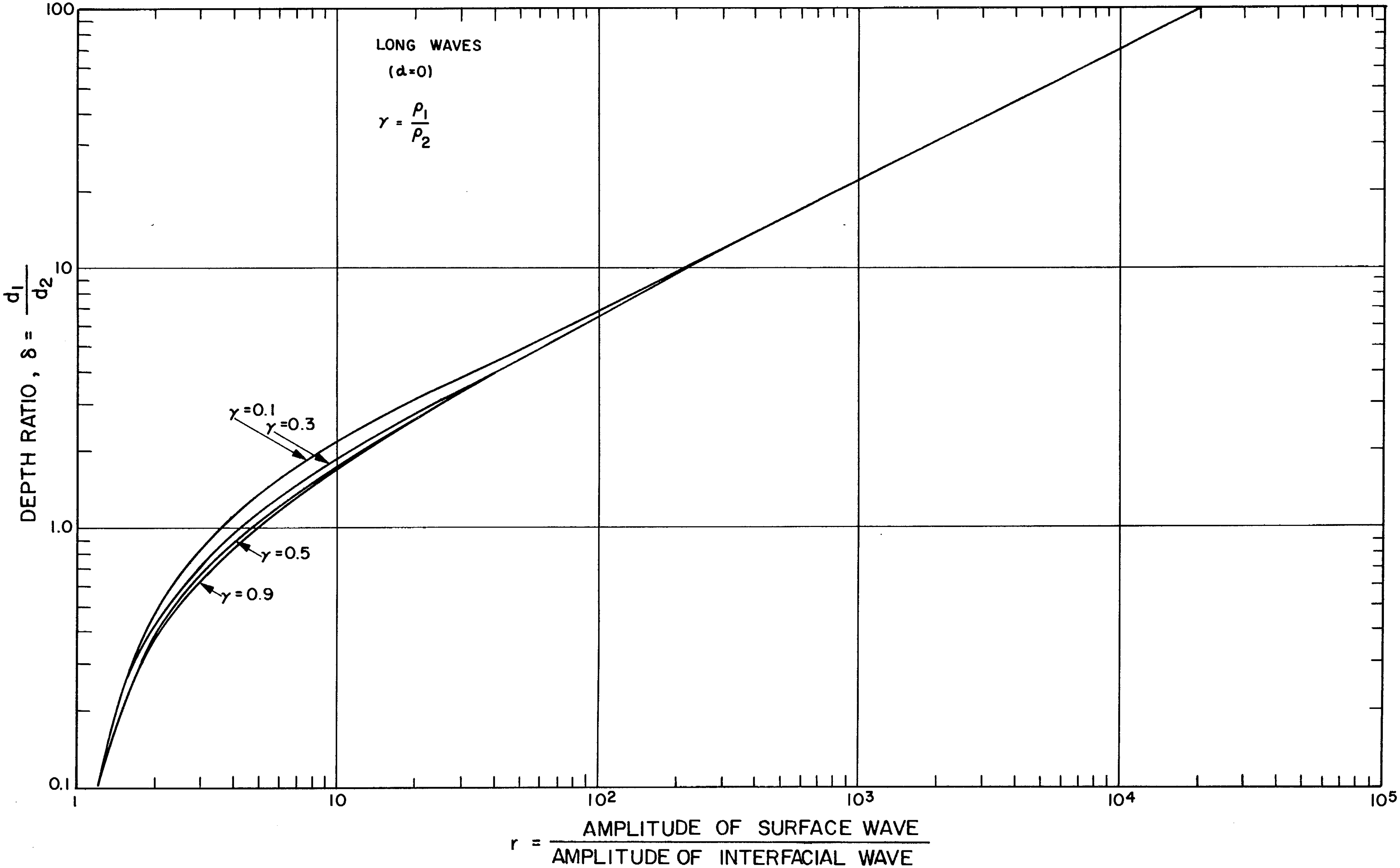

FIGURE 4

Fig. 4. Variation of amplitude ratio with depth ratio (δ) for constant density ratio (γ) for the limiting case of long waves.

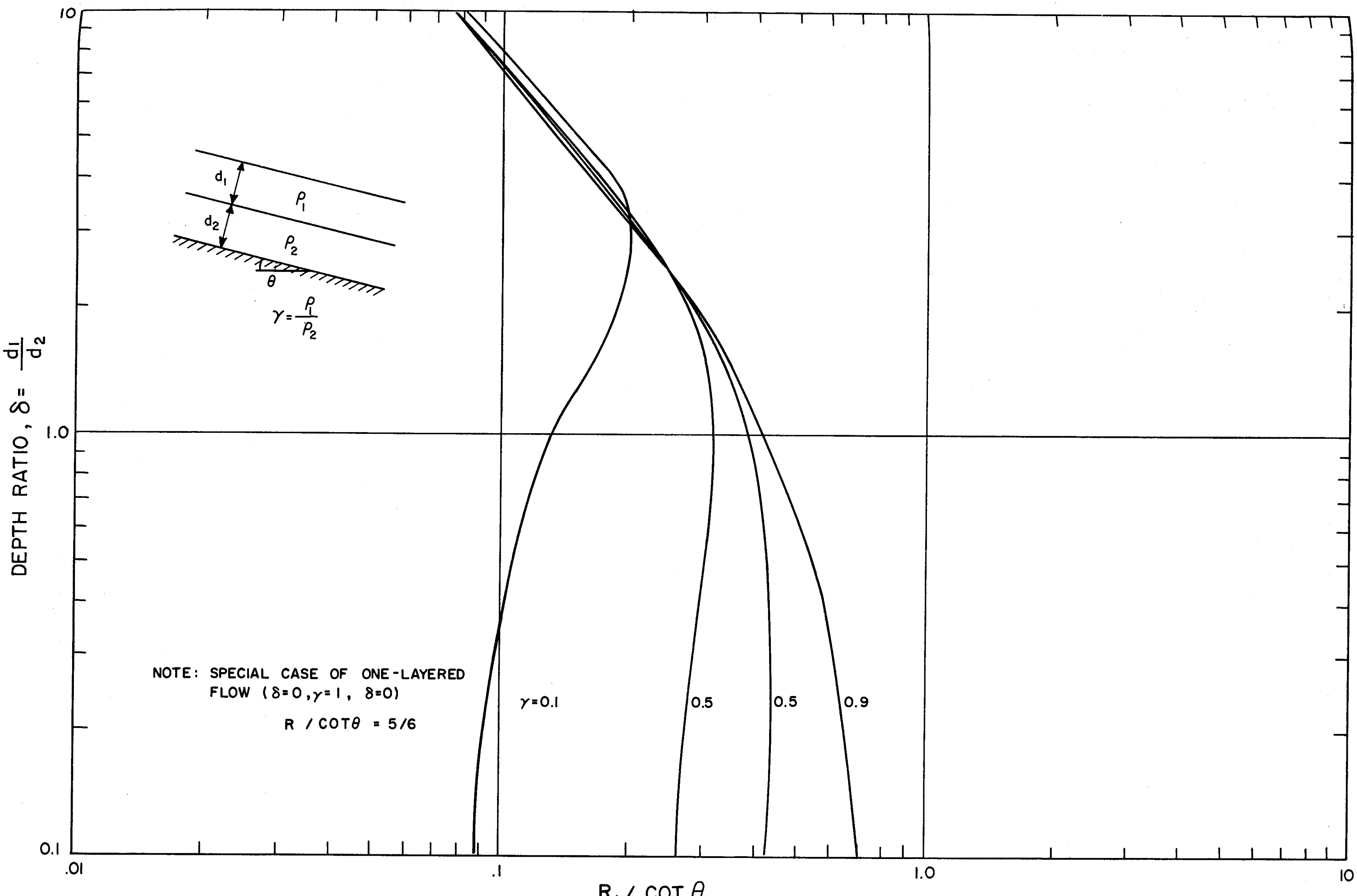

FIGURE 5

Fig. 5. Critical Reynolds number (R_1) as a function of depth ratio (δ) and density ratio (γ), based on computed bifurcation point of neutral curve on $\alpha = 0$.

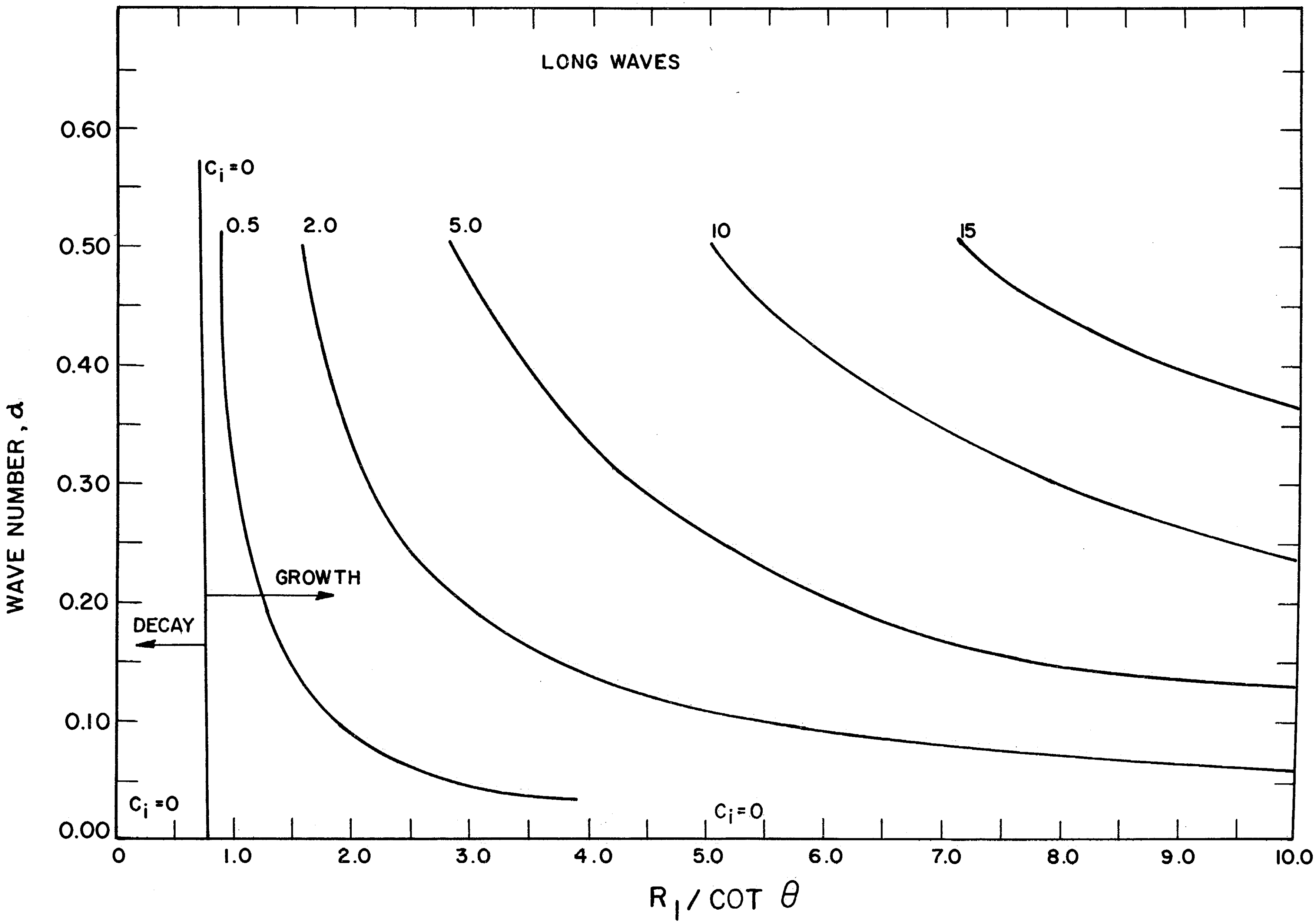

FIGURE 6

Fig. 6. Computed curves of constant c_i for $\gamma = 0.9$, $\delta = 1.0, \theta = 30°$, $S_1R_1 = S_2R_2 = 0.10$.

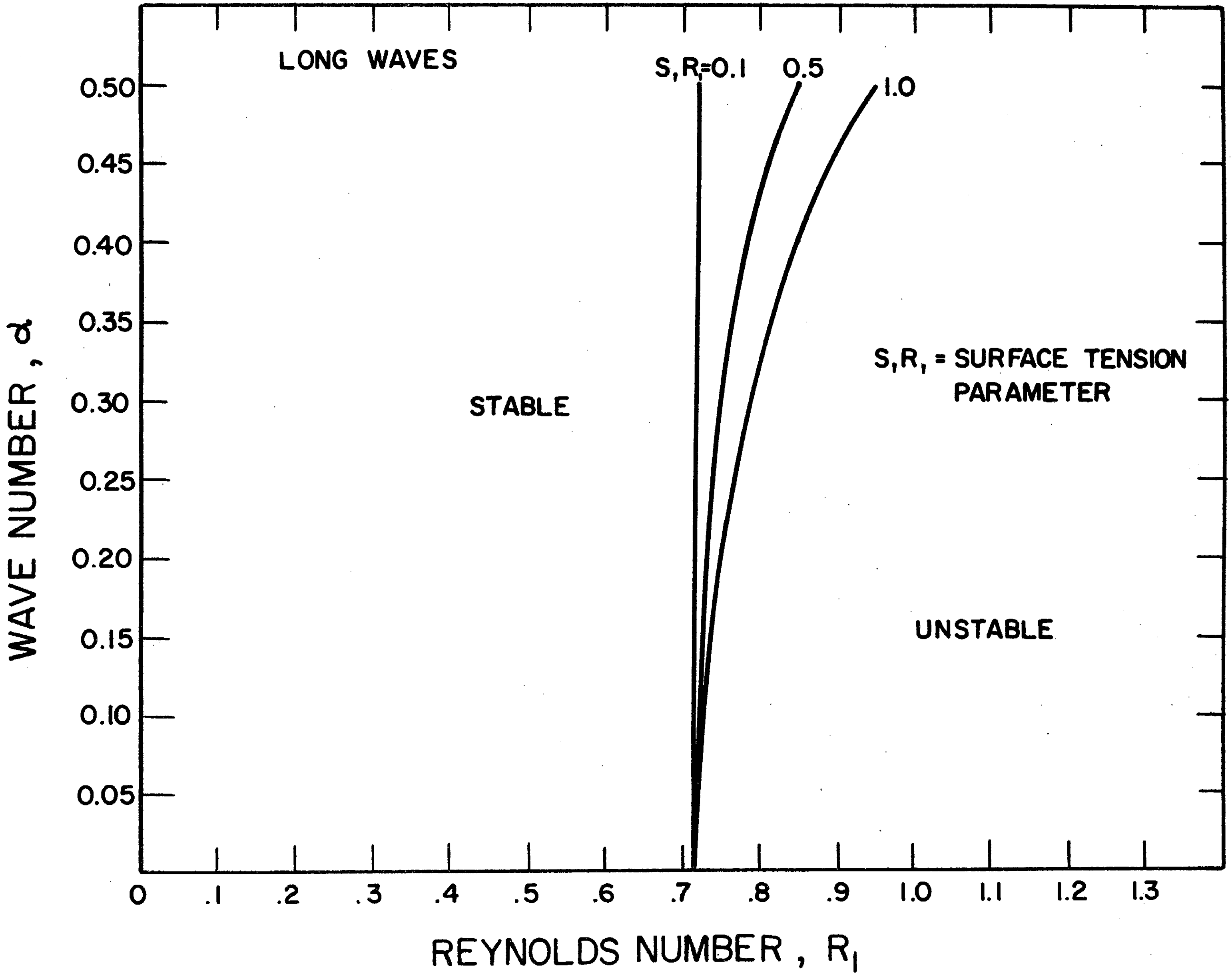

Fig. 7. Curves of neutral stability for various values of surface tension parameter, S_1R_1, for $\gamma = 0.9$, $\delta = 1.0$, $\theta = 30$.

ACKNOWLEDGEMENTS

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APPENDIX A

Derivation of Boundary Conditions

At the free surface, the kinematic condition is,

$$\frac{D(\xi - y)}{Dt} = 0 \quad , \tag{A 1}$$

Linearizing for small perturbations, we have,

$$\frac{\partial \xi}{\partial t} + U_1 \frac{\partial \xi}{\partial x} = v_1'$$

If we assume these perturbations are of the form,

$$\xi = a_1 \exp[i\alpha(x - ct)] \ , \tag{A2}$$

then we have,

$$a_1(-i\alpha c + i\alpha U_1) = -i\alpha \phi_1 \ ,$$

evaluated at y = $-\delta$. Hence,

$$a_1 = \frac{\phi_1(-\delta)}{c - U_1(-\delta)}$$

and

$$\xi(x, t) = \frac{\phi_1(-\delta)}{c - U_1(-\delta)} \exp[i\alpha(x - ct)] = \frac{\phi_1(-\delta)}{c_1} \exp[i\alpha(x - ct)] \tag{A3}$$

where

$$c_1 \equiv c - U_1(-\delta)$$

Similarly at the interface we have,

$$\eta(x,t) = \frac{\phi_2(0)}{c_2} \exp[i\alpha(x - ct)] \tag{A4}$$

where

$$c_2 \equiv c - U_2(0)$$

The dynamical conditions at the free surface are,

$$\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x} = 0 \quad , \tag{A5}$$

$$\left(-p_1 + \frac{2}{R_1}\frac{\partial v_1}{\partial y}\right) + S_1 \frac{\partial^2 \xi}{\partial x^2} = 0 \quad , \tag{A6}$$

where $S_1 = T_1/\rho_1 \bar{u}_a^2 d_2$, T_1 being the surface tension. Now expanding the basic flow quantities in Taylor's series, we have, from (A5)

$$\frac{\partial}{\partial y}\left[\, U_1(y) + U_1'(y)\xi + \ldots + u_1' \,\right] + \frac{\partial v'}{\partial x} = 0 \, ,$$

which on using (A3) and (A4), and evaluating y at $-\delta$, and realizing that at $y = -\delta$, $dU_1/dy = 0$, becomes

$$\frac{d^2 U_1}{dy^2} \cdot \frac{\phi_1(-\delta)}{c_1} + \phi_1''(-\delta) + \alpha^2 \phi_1(-\delta) = 0 \quad ,$$

But $\frac{d^2 U_1}{dy^2} = -K\gamma$ at $y = -\delta$. Hence (A5) becomes

$$\text{(i)} \qquad \phi_1''(-\delta) + \left(\alpha^2 - \frac{K\gamma}{c_1}\right)\phi_1(-\delta) = 0 \; .$$

From (A6) we have,

$$\left(-P_1 - \frac{dP_1}{dy}\xi - \ldots - p_1' - \frac{2}{R_1}\frac{\partial^2 \psi}{\partial x \partial y}\right) + S_1 \frac{\partial^2 \xi}{\partial x^2} = 0 \, ,$$

But $(P_1(-\delta) = 0$, and $\frac{dP_1}{dy} = \cos\theta / F^2$. Hence,

$$\frac{\cos\theta}{F^2}\frac{\phi_1(-\delta)}{c_1} - f_1(-\delta) - \frac{2i\alpha}{R}\phi_1'(-\delta) - \alpha^2 S_1 \frac{\phi_1(-\delta)}{c_1} = 0 \; .$$

or

$$-\alpha R_1 f_1(-\delta) - \alpha R_1 \left(\frac{\cos\theta}{F^2} + \alpha^2 S_1 \right) \frac{\phi_1(-\delta)}{c_1} - 2i\alpha^2 \phi_1'(-\delta) = 0$$

But from the first of the equations of motion (18) substituting (21) and (22), we have, on evaluating at y = -δ,

$$-\alpha R_1 f_1(-\delta) = -\alpha c_1 R_1 \phi_1'(-\delta) + i\, \phi_1'''(-\delta) - i\alpha^2 \phi_1'(-\delta) .$$

Hence eliminating $f_i(-\delta)$ from the above two equations and noting relationship given by (11), it follows that

(ii) $[\alpha(\gamma K \cot\theta + \alpha^2 S_1 R_1)/c_1]\, \phi_1(-\delta) + \alpha (R_1 c_1 + 3i\alpha)\phi_1'(-\delta) - i\, \phi_1'''(-\delta) = 0$

At the interface, the kinematic boundary conditions are,

$u_1' = u_2'$, $v_1' = v_2'$. Therefore,

(iii) $\phi_1(0) = \phi_2(0)$,

(iv) $\phi_1'(0) = \phi_2'(0)$.

The dynamical boundary conditions at the interface are,

$$\frac{\partial v_1}{\partial x} + \frac{\partial u_1}{\partial y} = \frac{\partial v_2}{\partial x} + \frac{\partial u_2}{\partial y} , \qquad \text{(A7)}$$

$$\left(-p_2 + \frac{2}{R_2}\frac{\partial v_2}{\partial y}\right) - \left(-p_1 + \frac{2}{R_1}\frac{\partial v_1}{\partial y}\right)\gamma + S_2 \frac{\partial^2 \eta}{\partial x^2} = 0 , \qquad \text{(A8)}$$

where $S_2 = T_2 / \rho_2 \bar{u}_a d_2$, T_2 being the surface tension from (A7), or expanding the basic flow quantities in Taylor's series, we have

$$\frac{d}{dy}\left[U_1 + U_1'\eta + U_1''\frac{\eta^2}{2} + \ldots \right] + \frac{\partial^2 \psi_1}{\partial y^2} - \frac{\partial^2 \psi_1}{\partial x^2} =$$

$$\frac{d}{dy}\left[U_2 + U_2'\eta + U_2''\frac{\eta^2}{2} + \ldots \right] + \frac{\partial^2 \psi_2}{\partial y^2} - \frac{\partial^2 \psi_2}{\partial x^2} .$$

But $\frac{dU_1}{dy} = \frac{dU_2}{dy}$, and $\varphi_1 = \varphi_2$, at $y = 0$. Therefore

$$[U_1''(0) - U_2''(0)]\frac{\phi_2(0)}{c_2} + \phi_1''(0) - \phi_2''(0) = 0 ,$$

Now $U_1''(0) - U_2''(0) = K(1-\gamma)$. It then follows that

(v) $$K(1-\gamma)\,\phi_2(0) + c_2\,\phi_1''(0) - c_2\,\phi_2''(0) = 0 .$$

Fron (A8), we have as before,

$$\left(-P_2 - P_2'\eta - p_2' - \frac{2}{R_2}\frac{\partial^2\psi_2}{\partial x\partial y}\right) + \gamma\left(P_1 + P_1'\eta + p_1' + \frac{2}{R_1}\frac{\partial^2\psi_1}{\partial x\partial y}\right) + S_2\frac{\partial^2\eta}{\partial x^2} = 0 .$$

Since $\gamma P_1 = P_2$ at $y=0$, and also $\frac{dP_1}{dy} = \cos\theta/F^2$, $\frac{dP_2}{dy} = \cos\theta/F^2$, thus on using (A4) we have

$$-\frac{\cos\theta}{F^2}(1-\gamma)\frac{\phi_2(0)}{c_2} - \left(f_2(0) - \gamma f_1(0)\right) + \frac{2i\alpha}{R_2}\left(\phi_2'(0) - \phi_1'(0)\right) - S_2\alpha^2\frac{\phi_2(0)}{c_2} = 0 ,$$

or

$$\left\{\left[(1-\gamma)\frac{\cos\theta}{F^2} + S_2\alpha^2\right]/c_2\right\}\phi_2(0) + \left[f_2(0) - \gamma f_1(0)\right] = 0 .$$

Again for the first of the equations of motion (18), we have on evaluating at $y = 0$,

$$f_i(0) = \left(c - U_i(0)\right)\phi_i'(0) + U_i'(0)\,\phi_i(0) + \frac{1}{i\alpha R_i}\left(\phi_i'''(0) - \alpha^2\phi_i'(0)\right) ,$$

where $i = 1$, or 2. From this it follows that

$$f_2(0) - \gamma f_1(0) = (1-\gamma)\left[c_2\,\phi_1'(0) - K\gamma\delta\,\phi_1(0)\right] + \frac{1}{i\alpha R_2}\left(\phi_2'''(0) - \phi_1'''(0)\right) .$$

Hence, it is found that,

(vi) $$c_2\left[\phi_2'''(0) - \phi_1'''(0)\right] + (1-\gamma)\,i\alpha R_2 c_2\left[c_2\,\phi_1'(0) - K\gamma\delta\,\phi_1(0)\right] + i\alpha\left[K(1-\gamma)\cot\theta + \alpha^2 S_2 R_2\right]\phi_1(0) = 0$$

The boundary conditions at the solid wall are that $u' = 0$, and $v' = 0$ at $y = 1$, i.e.

(vii) $\phi_2(1) = 0$,

(viii) $\phi_2'(1) = 0$.

APPENDIX B

Solution of Zeroth Order Approximation for Long Waves

With reference to equations (30) and (31) and the relevant boundaries conditions (i) through (viii) given below these equations, we have,

$$\phi_{10}(y) = A_1 + B_1 y + D_1 y^2 + E_1 y^3 ,$$

$$\phi_{20}(y) = A_2 + B_2 y + D_2 y^2 + E_2 y^3 .$$

(iii) $\longrightarrow$ $A_1 = A_2 \equiv A$,

(iv) $\longrightarrow$ $B_1 = B_2 \equiv B$,

(vi) $\longrightarrow$ $E_1 = E_2 \equiv E$,

(i) $\longrightarrow$ $2D_1 - 6\delta E - \frac{K\gamma}{c_1}[A - \delta B + D_1\delta^2 - \delta^3 E] = 0$,

(ii) $\longrightarrow$ $E = 0$,

(v) $\longrightarrow$ $K(1-\gamma)A + 2c_2 D_1 - 2c_2 D_2 = 0$,

(vii) $\longrightarrow$ $A + B + D_2 + E = 0 \longrightarrow A = -B - D_2$,

(viii) $\longrightarrow$ $B + 2D_2 + 3E = 0 \longrightarrow B = -2D_2$,

$\therefore \longrightarrow A = D_2$.

$\therefore$ (i) $\longrightarrow$ $2D_1 - \frac{K\gamma}{c_{10}}[D_2 + 2\delta D_2 + D_1\delta^2] = 0$,

(v) $\longrightarrow$ $K(1-\gamma)D_2 + 2c_{20}D_1 - 2c_{20}D_2 = 0$,

i.e.

$$(2c_{10} - K\gamma\delta^2)D_1 - K\gamma(1+2\delta)D_2 = 0 ,$$

$$2c_{20}D_1 + [K(1-\gamma) - 2c_{20}]D_2 = 0 .$$

In order that D_1, D_2, have nontrivial solutions, we must have,

$$\begin{vmatrix} (2c_{10} - K\gamma\delta^2) & -K\gamma(1+2\delta) \\ 2c_{20} & [K(1-\gamma) - 2c_{20}] \end{vmatrix} = 0 \quad ,$$

which, on expanding, yields,

$$c_{10}c_{20} - \frac{c_{10}}{2}K(1-\gamma) - \frac{c_{20}}{2}K\gamma(1+\delta)^2 + \frac{K^2\gamma\delta^2}{4}(1-\gamma) = 0 \quad .$$

Now $C_{10} = C_o - U_1(-\delta)$, and $C_{20} = C_o - U_2(0) = c_o - U_1(0)$. Hence, we have,

$$c_o^2 - \left[(U_1(-\delta) + U_1(0)) + \frac{K}{2}(\gamma(1+\delta)^2 + (1-\gamma)) \right] c_o + \left\{ U_1(-\delta)U_1(0) + \right.$$

$$\left. + \frac{K}{2}\left[U_1(-\delta)(1-\gamma) + U_1(0)\gamma(1+\delta)^2 + \frac{K\gamma\delta^2(1-\gamma)}{2} \right] \right\} = 0 \quad .$$

but,

$$U_1(-\delta) = \frac{K}{2}(1 + 2\gamma\delta + \gamma\delta^2) \, ,$$

$$U_1(0) = \frac{K}{2}(1 + 2\gamma\delta) \, ,$$

Therefore, solving the quadratic in C_o, we have

$$c_o = \frac{K}{4}(3 + 6\gamma\delta + 2\gamma\delta^2) \pm \frac{K}{2}\sqrt{\left(\frac{1}{4} + \gamma^2\delta^4 + 2\gamma^2\delta^3 + 3\gamma^2\delta^2 + \gamma\delta - \gamma\delta^2\right)} \, .$$

Since this is an eigenvalue problem D_1 or D_2 can be chosen arbitrarily. Choosing $D_2 = 1$, we then have,

$$D_1 = \frac{\gamma(1+2\delta)}{\frac{1}{2}(1 + 2\gamma\delta - 2\gamma\delta^2) \pm \sqrt{\left(\frac{1}{4} + \gamma^2\delta^4 + 2\gamma^2\delta^3 + 3\gamma^2\delta^2 + \gamma\delta - \gamma\delta^2\right)}} \quad .$$

Therefore, the eigenfunctions are,

$$\phi_{10} = \left(1 - 2y + \frac{\gamma(1+2\delta)}{\frac{1}{2}(1 + 2\gamma\delta - 2\gamma\delta^2) \pm \sqrt{\left(\frac{1}{4} + \gamma^2\delta^4 + 2\gamma^2\delta^3 + 3\gamma^2\delta^2 + \gamma\delta - \gamma\delta^2\right)}}\, y^2\right),$$

$$\phi_{20} = (1-y)^2 \, .$$

APPENDIX C

Solution of the First Order Approximation for Long Waves

The first order approximation for long waves is given by the set of equations (36), and (37) together with the boundary conditions, (i) through (viii) that followed these equations. Equations (36) and (37) are

$$\phi_{11}^{IV} = i\alpha R_1 \{ (U_1 - C_0)\, \phi_{10}'' - U_1''\, \phi_{10} \} ,$$

$$\phi_{21}^{IV} = i\alpha R_2 \{ (U_2 - C_0)\, \phi_{20}'' - U_2''\, \phi_{20} \} .$$

Now using the expressions for φ_{10}, φ_{20} given by equations (33), (34) and U_1, U_2 given by equations (12), (13) and Co given by equation (32), with the plus sign taken in front of the radical, the above two equations become

$$\phi_{11}^{IV} = i\alpha R_1 (-Ay + B) ,$$

$$\phi_{21}^{IV} = i\alpha R_2 \{ -2K(1+\gamma\delta)y + C \} ,$$

where

$$A = 2K\gamma + \frac{2K\gamma^2\delta(1+2\delta)}{\tfrac{1}{2}(1+2\gamma\delta-2\gamma\delta^2) + \sqrt{(\tfrac{1}{4}+\gamma^2\delta^4+2\gamma^2\delta^3+3\gamma^2\delta^2+\gamma\delta-\gamma\delta^2)}} ,$$

$$B = -\delta A ,$$

$$C = \frac{K}{2}(1-2\gamma\delta-2\gamma\delta^2) - K\sqrt{(\tfrac{1}{4}+\gamma^2\delta^4+2\gamma^2\delta^3+3\gamma^2\delta^2+\gamma\delta-\gamma\delta^2)} =$$

$$= B .$$

Therefore

$$\phi_{11} = -\frac{i\alpha R_1 A y^5}{120} + \frac{i\alpha R_1 B y^4}{24} + \Delta A_1 + \Delta B_1 y + \Delta D_1 y^2 + \Delta E_1 y^3 , \tag{C 1}$$

$$\phi_{21} = \frac{-2i\alpha R_2 K(1+\gamma\delta)y^5}{120} + \frac{i\alpha R_2 C y^4}{24} + \Delta A_2 + \Delta B_2 y + \Delta D_2 y^2 + \Delta E_2 y^3 . \tag{C2}$$

Substituting (C1) and (C2) into the eight boundary conditions, there results, the following set of equations:

$$-\frac{K\gamma}{c_{10}}\Delta A_1 + \frac{K\gamma\delta}{c_{10}}\Delta B_1 - \left(\frac{K\gamma\delta^2}{c_{10}} - 2\right)\Delta D_1 + \left(\frac{K\gamma\delta^3}{c_{10}} - 6\delta\right)\Delta E_1 = \frac{K\gamma}{c_{10}} i\alpha R_1\left(\frac{A\delta^5}{120} + \frac{B\delta^4}{24}\right) -$$
$$- i\alpha R_1\left(\frac{A\delta^3}{6} + \frac{B\delta^2}{2}\right) - \frac{K\gamma\Delta c}{(c_{10})^2}\,\phi_{10}(-\delta)\,, \tag{C3}$$

$$6\Delta E_1 = i\alpha R_1\left\{\left(\frac{A\delta^2}{2} + B\delta\right) - c_{10}\,\phi_{10}'(-\delta)\right\} - i\alpha\left\{\frac{(\gamma K\cot\theta + \alpha^2 S_1 R_1)\,\phi_{10}(-\delta)}{c_{10}}\right\}\,, \tag{C4}$$

$$\Delta A_1 - \Delta A_2 = 0\,, \tag{C5}$$

$$\Delta B_1 - \Delta B_2 = 0\,, \tag{C6}$$

$$K(1-\gamma)\,\Delta A_2 + 2c_{20}(\Delta D_1 - \Delta D_2) = -\Delta c\,(\phi_{10}''(0) - \phi_{20}''(0))\,, \tag{C7}$$

$$-\Delta E_1 + \Delta E_2 = -\frac{(1-\gamma)\,i\alpha R_2}{6}\left\{c_{20}\,\phi_{10}'(0) - K\gamma\delta\right\} - \frac{i\alpha}{6}\left\{\frac{K(1-\gamma)\cot\theta + \alpha^2 S_2 R_2}{c_{20}}\right\}, \tag{C8}$$

$$\Delta A_2 + \Delta B_2 + \Delta D_2 + \Delta E_2 = i\alpha R_2\left\{\frac{K(1+\gamma\delta)}{60} - \frac{c}{24}\right\}\,, \tag{C9}$$

$$\Delta B_2 + 2\Delta D_2 + 3\Delta E_2 = i\alpha R_2\left\{\frac{K(1+\gamma\delta)}{12} - \frac{c}{6}\right\}\,. \tag{C10}$$

We note that the determinant of the coefficients on the left hand side of the above equations is zero. Substitution of (C4) into (C3), and (C8) yields

$$-K\gamma\Delta A_1 + K\gamma\delta\Delta B_1 + (2c_{10} - K\gamma\delta^2)\Delta D_1 = i\alpha\Big[(6c_{10}\delta - K\gamma\delta^3)\Big\{\frac{R_1}{6}\left(\frac{A\delta^2}{2} + B\delta\right) -$$
$$-\frac{1}{6}\left(\frac{\gamma K\cot\theta + \alpha^2 S_1 R_1}{c_{10}}\,\phi_{10}(-\delta) + R_1 c_{10}\,\phi_{10}'(-\delta)\right)\Big\} - R_1\Big\{c_{10}\left(\frac{A\delta^3}{6} + \frac{B\delta^2}{2}\right) +$$
$$- K\gamma\left(\frac{A\delta^5}{120} + \frac{B\delta^4}{24}\right)\Big\}\Big] - \frac{K\gamma}{c_{10}}\Delta c\,\phi_{10}(-\delta)\,, \tag{C11}$$

$$\Delta E_2 = \frac{i\alpha R_1}{6}\left(\frac{A\delta^2}{2} + B\delta\right) - \frac{i\alpha}{6}\left\{\frac{K\gamma\cot\theta + \alpha^2 S_1 R_1}{c_{10}}\,\phi_{10}(-\delta) + R_1 c_{10}\,\phi_{10}'(-\delta)\right\} +$$
$$+ \frac{i\alpha}{6}\left\{-(1-\gamma)R_2\left[c_{20}\,\phi_{10}'(0) - K\gamma\delta\right] - \left[\frac{K(1-\gamma)\cot\theta + \alpha^2 S_2 R_2}{c_{20}}\right]\right\}\,. \tag{C12}$$

Substituting of (C 12) into (C 9) and (C 10) yields

$$\Delta A_2 + \Delta B_2 + \Delta D_2 = i\alpha R_2 \left\{ \frac{K(1+\gamma\delta)}{60} - \frac{C}{24} \right\} - \frac{i\alpha R_1}{6} \left(\frac{A\delta^2}{2} + B\delta \right) + + \frac{i\alpha}{6} \left\{ \frac{K\gamma \cot\theta + \alpha^2 S_1 R_1}{c_{10}} \phi_{10}(-\delta) + R_1 c_{10} \phi_{10}'(-\delta) \right\} + \frac{i\alpha}{6} \left\{ (1-\gamma) R_2 \left[c_{20} \phi_{10}'(0) - K\gamma\delta \right] + \left[\frac{K(1-\gamma)\cot\theta + \alpha^2 S_2 R_2}{c_{20}} \right] \right\}, \tag{C 13}$$

$$\Delta B_2 + 2\Delta D_2 = i\alpha R_2 \left(\frac{K(1+\gamma\delta)}{12} - \frac{C}{6} \right) - \frac{i\alpha R_1}{2} \left(\frac{A\delta^2}{2} + B\delta \right) + \frac{i\alpha}{2} \left\{ \frac{K\gamma\cot\theta + \alpha^2 S_1 R_1}{c_{10}} \phi_{10}(-\delta) + R_1 c_{10} \phi_{10}'(-\delta) \right\} + \frac{i\alpha}{2} \left\{ (1-\gamma) R_2 \left[c_{20} \phi_{10}'(0) - K\gamma\delta \right] + \left[\frac{K(1-\gamma)\cot\theta + \alpha^2 S_2 R_2}{c_{20}} \right] \right\}. \tag{C 14}$$

In order to simplify writing we let,

$$b_1 = R_1 \left\{ (6c_{10}\delta - K\gamma\delta^3) \left[\frac{\left(\frac{A\delta^2}{2} + B\delta\right)}{6} - \frac{c_{10}\phi_{10}'(-\delta)}{6} \right] - \left[c_{10} \left(\frac{A\delta^3}{6} + \frac{B\delta^2}{2} \right) - - K\gamma \left(\frac{A\delta^5}{120} + \frac{B\delta^4}{24} \right) \right] \right\} - \left\{ \frac{(6c_{10}\delta - K\gamma\delta^3)}{6} \left[\frac{K\gamma\cot\theta + \alpha^2 S_1 R_1}{c_{10}} \phi_{10}(-\delta) \right] \right\}, \tag{C 15}$$

$$b_2 = R_1 \left\{ \frac{1}{\gamma} \left(\frac{K(1+\gamma\delta)}{60} - \frac{C}{24} \right) - \frac{1}{6} \left(\frac{A\delta^2}{2} + B\delta \right) + \frac{1}{6} c_{10} \phi_{10}'(-\delta) + \frac{(1-\gamma)}{6\gamma} \left(c_{20}\phi_{10}'(0) - - K\gamma\delta \right) \right\} + \frac{1}{6} \left\{ \frac{K\gamma\cot\theta + \alpha^2 S_1 R_1}{c_{10}} \phi_{10}(-\delta) + \frac{K(1-\gamma)\cot\theta + \alpha^2 S_2 R_2}{c_{20}} \right\}, \tag{C 16}$$

$$b_3 = R_1 \left\{ \frac{1}{\gamma} \left(\frac{K(1+\gamma\delta)}{12} - \frac{C}{6} \right) - \frac{1}{2} \left(\frac{A\delta^2}{2} + B\delta \right) + \frac{1}{2} c_{10} \phi_{10}'(-\delta) + \frac{(1-\gamma)}{2\gamma} \left(c_{20}\phi_{10}'(0) - - K\gamma\delta \right) \right\} + \frac{1}{2} \left\{ \frac{K\gamma\cot\theta + \alpha^2 S_1 R_1}{c_{10}} \phi_{10}(-\delta) + \frac{K(1-\gamma)\cot\theta + \alpha^2 S_2 R_2}{c_{20}} \right\}. \tag{C 17}$$

Using (C5) and (C6) and eliminating ΔA_2, and ΔB_2 from (C 11), (C 12), (C 13), and (C 14), we obtain,

$$(2c_{10} - K\gamma\delta^2)\,\Delta D_1 \quad - K\gamma\,(1+2\delta)\Delta D_2 \;=\; i\alpha\left\{b_1 + K\gamma(b_2-b_3) - K\gamma\delta b_3\right\} - \frac{K\gamma}{c_{10}}\Delta c\,\phi_{10}(-\delta)\,, \qquad (C18)$$

$$2c_{20}\,\Delta D_1 \;+\; \left\{K(1-\gamma) - 2c_{20}\right\}\Delta D_2 \;=\; i\alpha\,(b_2-b_3)\,K(1-\gamma) + \Delta c\left\{\phi_{20}''(0) - \phi_{10}''(0)\right\}. \qquad (C19)$$

But
$$\begin{vmatrix} (2c_{10} - K\gamma\delta^2) & -K\gamma(1+2\delta) \\ 2c_{20} & \left\{K(1-\gamma) - 2c_{20}\right\} \end{vmatrix} = 0\,,$$

Therefore, we have,

$$-i\alpha(b_2-b_3)\,K(1-\gamma) + \Delta c\left\{\phi_{20}''(0) - \phi_{10}''(0)\right\} = \frac{2c_{20}}{(2c_{10}-K\gamma\delta^2)}\left\{i\alpha\left[b_1 + K\gamma(b_2-b_3) - K\gamma\delta b_3\right] - \frac{K\gamma}{c_{10}}\Delta c\;\phi_{10}(-\delta)\right\}. \qquad (C20)$$

From which we have,

$$\Delta c = i\alpha\left\{\frac{2c_{10}c_{20}\left[b_1 + K\gamma(b_2-b_3) - K\gamma\delta b_3\right] + c_{10}(b_2-b_3)K(1-\gamma)(2c_{10}-K\gamma\delta^2)}{c_{10}(2c_{10}-K\gamma\delta^2)\left(\phi_{20}''(0)-\phi_{10}''(0)\right) + 2c_{20}\gamma K\,\phi_{10}(-\delta)}\right\}. \qquad (C21)$$

which on collecting terms, yields,

$$\Delta c = i\alpha\left\{\frac{G}{H}R_1 - \frac{1}{H}\left(\phi\cot\theta + \lambda\alpha^2\right)\right\}.$$

as it appears in equation (38).

APPENDIX D

Secular equation for the case of long shear waves

From (42') and (43) and substituting into the boundary conditions listed below equations (40) and (41) we obtain:

$$C_1\beta^2 e^{-\beta\delta} + D_1\beta^2 e^{\beta\delta} = 0 \quad , \tag{D1}$$

$$\beta^2(B_1 + C_1\beta e^{-\beta\delta} - D_1\beta e^{\beta\delta}) - (C_1\beta^3 e^{-\beta\delta} - D_1\beta^3 e^{\beta\delta}) = 0 \quad , \tag{D2}$$

$$A_1 + C_1 + D_1 = A_2 + C_2 + D_2 \quad , \tag{D3}$$

$$B_1 + C_1\beta - D_1\beta = B_2 + C_2\frac{\beta}{\sqrt{\gamma}} - D_2\frac{\beta}{\sqrt{\gamma}} \quad , \tag{D4}$$

$$C_1\beta^2 + D_1\beta^2 = C_2\frac{\beta^2}{\gamma} + D_2\frac{\beta^2}{\gamma} \quad , \tag{D5}$$

$$\left(C_2\frac{\beta^3}{\gamma^{3/2}} - D_2\frac{\beta^3}{\gamma^{3/2}}\right) - (C_1\beta^3 - D_1\beta^3) - \left(\frac{1-\gamma}{\gamma}\right)\beta^2(B_1 + C_1\beta - D_1\beta) = 0 , \tag{D6}$$

$$A_2 + B_2 + C_2 e^{\beta/\sqrt{\gamma}} + D_2 e^{-\beta/\sqrt{\gamma}} = 0 \quad , \tag{D7}$$

$$B_2 + C_2\frac{\beta}{\sqrt{\gamma}} e^{\beta/\sqrt{\gamma}} - D_2\frac{\beta}{\sqrt{\gamma}} e^{-\beta/\sqrt{\gamma}} = 0 \quad . \tag{D8}$$

$$(D1) \longrightarrow \quad C_1 = -e^{2\beta\delta} D_1 \tag{D9}$$

$$(D2) + (D1) \longrightarrow \quad B_1 = 0 \tag{D10}$$

Substitution of (D9) and (D10) into (D3) through (D8), yields,

$$A_1 - D_1(e^{2\beta\delta} - 1) = A_2 + C_2 + D_2 \quad , \tag{D11}$$

$$-\beta(e^{2\beta\delta} + 1)D_1 = B_2 + \frac{\beta}{\sqrt{\gamma}}C_2 - \frac{\beta}{\sqrt{\gamma}}D_2 \quad , \tag{D12}$$

$$-\beta^2(e^{2\beta\delta} - 1)D_1 = \frac{\beta^2}{\gamma}(C_2 + D_2) , \tag{D13}$$

$$\frac{1}{\gamma^{3/2}}(C_2 - D_2) + (e^{2\beta\delta} + 1)D_1 + \left(\frac{1-\gamma}{\gamma}\right)(e^{2\beta\delta} + 1)D_1 = 0 , \tag{D14}$$

$$A_2 + B_2 + C_2 e^{\beta/\sqrt{\gamma}} + D_2 e^{-\beta/\sqrt{\gamma}} = 0 \quad , \tag{D15}$$

$$B_2 + C_2\frac{\beta}{\sqrt{\gamma}} e^{\beta/\sqrt{\gamma}} - D_2\frac{\beta}{\sqrt{\gamma}} e^{-\beta/\sqrt{\gamma}} = 0 \quad . \tag{D16}$$

$$(D14) \longrightarrow \quad (C_2 - D_2) = -\sqrt{\gamma}(e^{2\beta\delta} + 1)D_1 \quad , \tag{D17}$$

$$(D12) + (D17) \longrightarrow \quad B_2 = 0 \quad , \tag{D18}$$

$$(D13) \longrightarrow \quad (C_2 + D_2) = -\gamma(e^{2\beta\delta} - 1)D_1 \quad , \tag{D19}$$

From (D17) and (D19), expressing C_2, D_2 in terms of D_1, we have

$$C_2 = \tfrac{1}{2}\left\{-(\gamma+\sqrt{\gamma})\,e^{2\beta\delta} + (\gamma-\sqrt{\gamma})\right\} D_1 \,, \tag{D20}$$

$$D_2 = \tfrac{1}{2}\left\{-(\gamma-\sqrt{\gamma})\,e^{2\beta\delta} + (\gamma+\sqrt{\gamma})\right\} D_1 \,. \tag{D21}$$

Putting the values found for B_2, C_2, and D_2, into (D16), there results the secular equation

$$\cosh\frac{\beta}{\sqrt{\gamma}}\cosh\beta\delta + \sqrt{\gamma}\,\sinh\frac{\beta}{\sqrt{\gamma}}\sinh\beta\delta = 0\,, \qquad \gamma > 0\,.$$

APPENDIX E

Expansion of the Determinantal Equation for the solution of short waves.

The determinantal equation (50) is expanded by the following successive steps.

(1) Multiply column (1) by δ and add to it column (3).

(2) Multiply column (2) by δ and add to it column (4).

(3) Add to column (2) column (1) and then divide by 2.

(4) Subtract from column (5), column (6).

(5) Subtract from column (7), column (8).

(6) Subtract column (2) from column (1).

(7) Multiply column (6) by δ and add column (2) to it.

(8) Factorize from Row (4) c_2

(9) Factorize from Row (5) αc_2

(10) Multiply Row (1) by α and add to it Row (2).

The determinant is now expanded by Laplace's method, yielding the following expression,

$$\frac{2\alpha^2 c_1 (\cosh\alpha\delta)\, e^{-\alpha\delta}}{c_1}\Big\{-K(1-\gamma)\alpha^3 e^{-\alpha}(\sinh\alpha+\cosh\alpha) + 3\alpha^3 e^{-2\alpha}\delta c_2 +$$

$$+ 2\alpha\left(i\delta N - (3\alpha+\alpha^2\delta)c_2\right)e^{-\alpha}\left[(1-\alpha)\sinh\alpha - \alpha\cosh\alpha\right]\Big\} +$$

$$+ \frac{2\alpha^4 c_1 \cosh\alpha\delta\, e^{-3\alpha\delta}}{c_2}\Big\{2\alpha\left(i\delta N - (3\alpha+\alpha^2\delta)c_2\right)(\sinh\alpha+\cosh\alpha) -$$

$$- 3\alpha K(1-\gamma)(\sinh\alpha+\cosh\alpha) + 2\delta e^{-\alpha\delta}(1-2\alpha)c_2\Big\} -$$

$$- 4\alpha^5 c_1 (\cosh\alpha\delta)\, e^{-2\alpha\delta} (\alpha\delta+1)(\alpha+3)(\sinh\alpha + \cosh\alpha) -$$

$$- \frac{\delta \cosh\alpha\delta\, (iM + 2\alpha^2 c_1) \sinh\alpha\delta}{c_2} \Big\{ -K(1-\gamma)\, 4\alpha^3 e^{-\alpha} (\sinh\alpha + \cosh\alpha) +$$

$$+ 8\alpha^3 \delta e^{-2\alpha} c_2 + 2\alpha e^{-\alpha} [((1-\alpha)\sinh\alpha - \alpha\cosh\alpha)(i\delta N + 2\alpha^2 \delta c_2) -$$

$$- 4\alpha^3 \delta e^{-\alpha} [(1+\alpha)\sinh\alpha + \alpha\cosh\alpha]\, c_2 - 2\alpha \sinh^2\alpha\, (i\delta N + 2\alpha^2\delta c_2) \Big\} +$$

$$+ \frac{2\alpha\delta \cosh^2\alpha\delta\, (iM + 2\alpha^2 c_1)}{c_2} \Big\{ \alpha^2 e^{-\alpha} (\sinh\alpha + \cosh\alpha)(i\delta N + 2\alpha^2 \delta c_2) -$$

$$- 4\alpha^3 (\alpha\delta+1) e^{-\alpha} (\sinh\alpha + \cosh\alpha) c_2 + 4\alpha^2 \delta e^{-\alpha} [(1+\alpha)\sinh\alpha +$$

$$+ \alpha\cosh\alpha]\, c_2 + \sinh^2\alpha\, (i\delta N + 2\alpha^2\delta c_2) \Big\} - \frac{\delta\, [iM(1-\alpha) + \alpha K\gamma]}{c_2} \Big\{$$

$$\alpha\delta \sinh\alpha\, (i\delta N - 2\alpha c_2)(-2\alpha(1-\alpha) e^{-\alpha} - 2\alpha \sinh\alpha - 2\alpha^2\cosh\alpha) -$$

$$- 2\alpha^2\delta \sinh\alpha\, [K(1-\gamma)(\sinh\alpha + \alpha\cosh\alpha) - 2\alpha^2\delta e^{-\alpha} c_2] -$$

$$- 2\alpha^3\delta \cosh\alpha\, (i\delta N - 2\alpha c_2)(e^{-\alpha} + \sinh\alpha) + 2\alpha^3\delta \cosh\alpha\, [K(1-\gamma)\sinh\alpha +$$

$$+ 2\alpha\delta e^{-\alpha} c_2] - 2\alpha^3 (i\delta N - 2\alpha c_2) e^{-\alpha} (\sinh\alpha + \cosh\alpha) +$$

$$+ 4\delta\alpha^3 e^{-\alpha} [(1+\alpha)\sinh\alpha + \alpha\cosh\alpha]\, c_2 + 2\alpha \sinh^2\alpha\, (2\alpha^2\delta c_2 + i\delta N) \Big\} +$$

$$+ \delta\alpha^4 \sinh\alpha\delta\, e^{-\alpha(1+\delta)} c_1 \Big\{ \alpha^2 (\sinh\alpha + \cosh\alpha) + 3(\alpha\delta+2)[(1-\alpha)\sinh\alpha -$$

$$- \alpha\cosh\alpha)] \Big\} - \delta \sinh\alpha\delta\, (iM + 2\alpha^2 c_1) \Big\{ 4\alpha^3\delta \sinh^2\alpha\, (\sinh\alpha\delta -$$

$$- \cosh\alpha\delta) + 4\alpha^3 \delta \sinh\alpha\delta\, e^{-\alpha} [(1-\alpha)\sinh\alpha - \alpha\cosh\alpha] -$$

$$- 4\alpha^4 \sinh\alpha\delta\, e^{-\alpha} (\sinh\alpha + \cosh\alpha) \Big\} = 0$$

in which $\delta > 0$, $\alpha \gg 1$.

As $\alpha \rightarrow \infty$, the dominant term of the above equation is

$$\frac{\cosh \alpha\delta \sinh \alpha\delta \,(iM + 2\alpha^2 C_1)}{C_2} \left\{ 2\alpha \sinh^2 \alpha \,(i\delta N + 2\alpha^2 \delta C_2) \right\} +$$

$$+ \frac{2\alpha \cosh^2 \alpha\delta \,(iM + 2\alpha^2 C_1)}{C_2} \left\{ \sinh^2 \alpha \,(i\delta N + 2\alpha^2 \delta C_2) \right\} = 0$$

or,

$$2\alpha\delta \cosh \alpha\delta \sinh^2 \alpha\delta \,(\sinh \alpha\delta + \cosh \alpha\delta)\,(i\alpha^2 S_1 R_1 + 2\alpha^2 C_1) .$$

$$(i\alpha^2 S_2 R_2 + 2\alpha^2 C_2) = 0$$

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ADDENDUM

STABILITY OF TWO-LAYER STRATIFIED FLOW DOWN AN INCLINED PLANE

by

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A more meaningful question to ask with respect to the relative stability of the various flow configurations is this: For the same total depth, how does the stability of flow with stratification compare with the homogeneous case?# We can answer this question by defining a relative stability index, s, as follows:

$$s = \frac{\text{critical depth for two-layer flow for a given } \theta.}{\text{critical depth for homogeneous flow for same } \theta.}$$

If $s < 1$, the two-layer flow is more unstable than the homogeneous flow. Indeed, if a flow of a homogeneous fluid of depth h is critical, then, when $s<1$, the replacement of the homogeneous fluid by one with two layers of the same total depth h will make the flow unstable. If $s > 1$, the situation is reversed.

From the definition of the Reynolds number R_1, the critical depth for two-layer flow is given by

$$\left[\frac{K R_{1cr} \mu^2 (1+\delta)^3}{\gamma \rho_2^2 g \sin\theta} \right]^{1/3},$$

and the critical depth for a homogeneous flow is

$$\left[\frac{3 R_{cr} \mu^2 (1+\delta)}{\rho_2^2 g \sin\theta} \right]^{1/3}$$

Therefore

$$s = \left[\frac{K R_{1cr} (1+\delta)^2}{3 R_{cr} \gamma} \right]^{1/3}$$

But, since the total depth of flow is $d_2(1+\delta)$,

$$R_{cr}(1+\delta) = 5/6$$

Hence, it follows that

$$s = 0.737\,(1+\delta)\left(\frac{K R_{1cr}}{\gamma}\right)^{1/3}$$

Figure 8 gives the plot of the relative stability index s against the ratio of depths for various values of the ratio of density. It is to be noted that $\gamma = 1$ gives a constant $s = 1$ as it should be of course, and this line marks the region of relative stability and instability. It is seen that if the density of the upper layer is smaller than that of the lower layer, the effect of stratification is to make the flow more stable. This confirms our intuitive idea of the stabilizing effect of stratification of this kind. On the other hand, if the upper layer is of higher density than the lower fluid, the flow is more unstable than the homogeneous fluid.

The stability is now actually governed by the location of the interface and the ratio of the density. The potential energy required to distort the interface becomes smaller and smaller as the ratio of density becomes higher and higher and hence the flow becomes more and more unstable. Hence the more the difference in density, the more the stabilizing or destabilizing effect depending on whether ρ_1 is less than or greater than ρ_2. These arguments are very clearly borne out by the calculations and can be seen from this figure.

#The author is indebted to Professor C.-S. Yih of the University of Michigan for posing this question in a private communication.

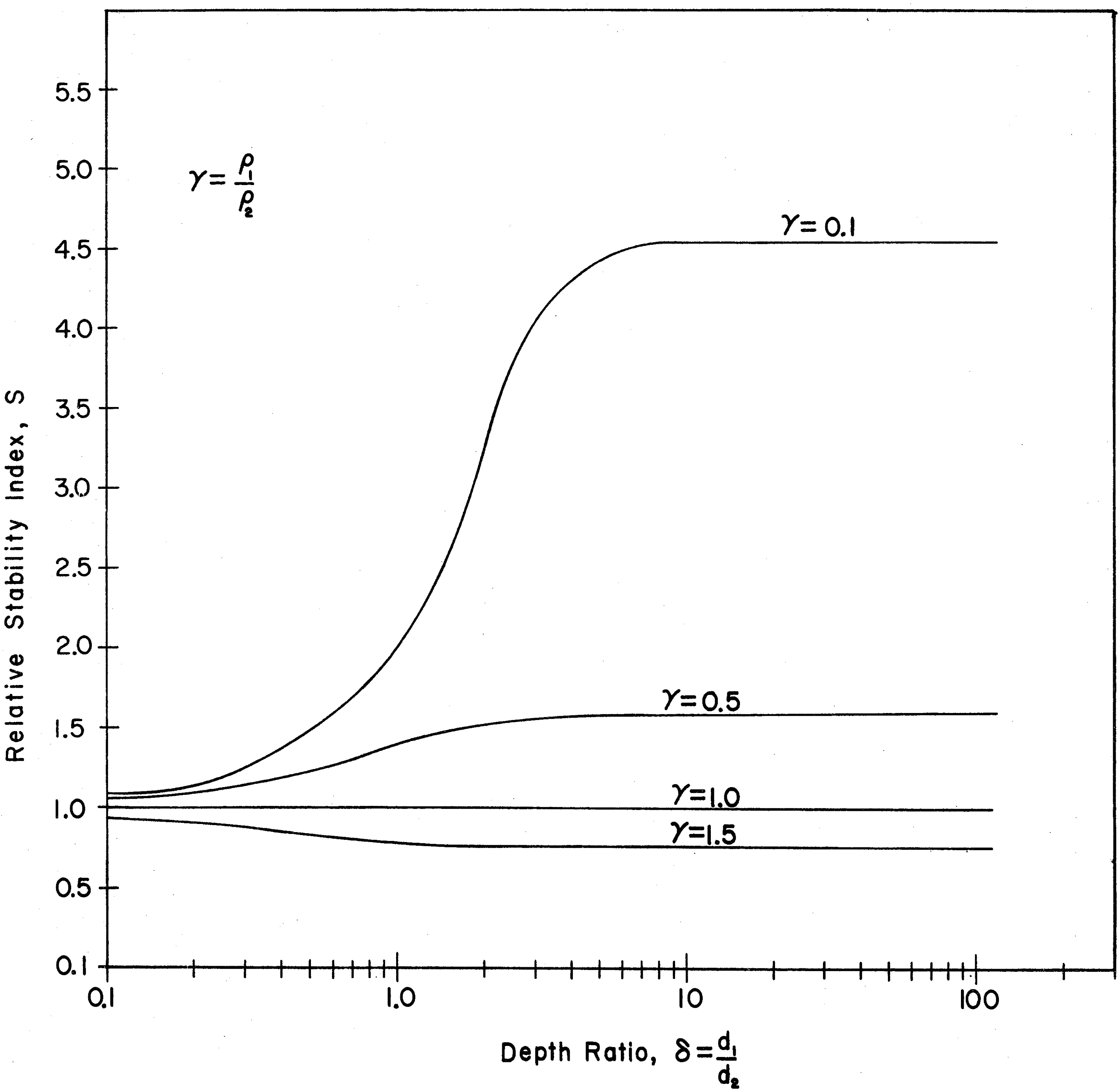

Fig. 8. Relative Stability Index as a function of depth ratio and density ratio